

# Averaging principle for one dimensional stochastic Burgers equation

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**Abstract.** This paper considers the averaging principle for one dimensional stochastic Burgers equation with slow and fast time-scales. Under some suitable conditions, we show that the slow component strongly converges to the solution of the corresponding averaged equation. Meanwhile, when there is no noise in the slow equation, we also prove the slow component weakly converges to the solution of the corresponding averaged equation with the convergent order of  $1 - r$ , for any  $0 < r < 1$ .

**Keywords:** Stochastic Burger's equation; Averaging principle; Ergodicity; Invariant measure; Strong convergence; Weak convergence

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## 1 Introduction

Almost all physical systems have a certain hierarchy in which not all components evolve at the same rate, i.e., some of components vary very rapidly, while others change very slowly, see [21]. Averaging principle provides an effective tool to analyze the multiple time-scales dynamical systems. Besides, the method of averaging principle enormously reduces the computational load although computer technology is highly efficient nowadays. The Michaelis-Menten approximation technique for the enzyme activation reactions [20] is a famous and successful example on this topic.

The theory of averaging principle has a long and rich history, which has been applied in many fields, such as, celestial mechanics, wireless communication, signal processing, oscillation theory and radiophysics. The first result for deterministic systems was studied by Bogoliubov [1], then by Volosov [24] for ordinary differential equations. In 1968, the theory of averaging principle for stochastic differential equations driven by Brownian motion was firstly proved by Khasminskii [17]. Since then, averaging principle for stochastic

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reaction-diffusion systems has become an active research area which attracted a number of mathematicians. For example, Freidlin and Wentzell [11] established the theory of averaging principle from a deeper understanding of this phenomenon. Cerrai and Freidlin [4] proved the averaging principle for a general class of stochastic reaction-diffusion systems, which extended the classical Khasminskii-type averaging principle for finite dimensional systems to infinite dimensional systems. We refer to [2, 3, 12, 13, 14, 15, 18, 19, 23] and references therein for more interesting results on this topic.

In this paper, we consider the averaging principle for one dimensional stochastic Burgers equation. To the best of our knowledge, this is the first article to deal with highly nonlinear term on this topic.

Considering the following stochastic fast-slow system on a bounded interval  $[0, 1] \subset \mathbb{R}$ :

$$\begin{cases} \frac{\partial X_t^\varepsilon(\xi)}{\partial t} = [\Delta X_t^\varepsilon(\xi) + \frac{1}{2} \frac{\partial}{\partial \xi} (X_t^\varepsilon(\xi))^2 + f(X_t^\varepsilon(\xi), Y_t^\varepsilon(\xi))] + \frac{\partial W^{Q_1}}{\partial t}(t, \xi), & X_0^\varepsilon = x \\ \frac{\partial Y_t^\varepsilon(\xi)}{\partial t} = \frac{1}{\varepsilon} [\Delta Y_t^\varepsilon(\xi) + g(X_t^\varepsilon(\xi), Y_t^\varepsilon(\xi))] + \frac{1}{\sqrt{\varepsilon}} \frac{\partial W^{Q_2}}{\partial t}(t, \xi), & Y_0^\varepsilon = y \\ X_t^\varepsilon(0) = X_t^\varepsilon(1) = Y_t^\varepsilon(0) = Y_t^\varepsilon(1) = 0, \end{cases} \quad (1.1)$$

where  $\varepsilon$  is a small and positive parameter describing the ratio of time scale between the slow component  $X_t^\varepsilon$  and fast component  $Y_t^\varepsilon$ . The coefficients  $f$  and  $g$  are suitable functions.  $\{W_t^{Q_1}\}_{t \geq 0}$  and  $\{W_t^{Q_2}\}_{t \geq 0}$  are  $H(= L^2(0, 1))$ -valued mutually independent  $Q_1$  and  $Q_2$ -Wiener processes on complete probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  with a filtration  $\{\mathcal{F}_t, t \geq 0\}$  satisfying the usual conditions.

When  $\varepsilon$  goes to 0, the slow component  $X^\varepsilon$  tends to process  $\bar{X}$ , which is the solution of the averaged equation:

$$\begin{cases} d\bar{X}_t = \Delta \bar{X}_t dt + \frac{1}{2} \frac{\partial}{\partial \xi} (\bar{X}_t)^2 dt + \bar{f}(\bar{X}_t) dt + dW^{Q_1}(t), \\ \bar{X}_0 = x. \end{cases} \quad (1.2)$$

with average  $\bar{f}(x) = \int_H f(x, y) \mu^x(dy)$ , where  $\mu^x$  denotes the unique invariant measure for the fast motion equation when we fix slow variable  $x \in H$  ( see equation (4.18) below ).

We aim to study the speed of  $X^\varepsilon$  convergent to  $\bar{X}$  in the strong and weak sense respectively. Under some conditions, the result of strong convergence is stated as follows:

- For any initial value  $x \in H^\alpha$  with  $\alpha \in (1, \frac{3}{2}]$  and  $y \in H$ ,  $p, T > 0$ , we have

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E} \sup_{0 \leq t \leq T} |X_t^\varepsilon - \bar{X}_t|^{2p} \leq C \left( \frac{1}{-\log \varepsilon} \right)^{\frac{1}{4p}} \rightarrow 0 \quad (\varepsilon \rightarrow 0). \quad (1.3)$$

where  $C$  is a positive constant which only depends on  $T$ ,  $p$ ,  $|x|_\alpha$  and  $|y|$ .

Furthermore, if there is no noise (i.e.,  $Q_1 = 0$ ) in system (1.1), under some conditions, the result of weak convergence is stated as follows:

- For any initial value  $x \in H^\theta$  with  $\theta \in (0, 1]$ ,  $y \in H$ ,  $\phi \in C_b^2$ ,  $r \in (0, 1)$ ,  $\delta \in (0, \frac{1}{2})$ , we have

$$|\mathbb{E}[\phi(X^\varepsilon(t))] - \mathbb{E}[\phi(\bar{X}(t))]| \leq C(1 + t^{-\theta + \frac{\theta^2}{1+\delta}}) \varepsilon^{1-r}, \quad (1.4)$$

where  $C$  is a positive constant which only depends on  $T$ ,  $|x|_\theta$ ,  $|y|$ ,  $\delta$  and  $\phi$ .

Compared with the strong convergence, the regularity of initial value  $x$  in weak convergence is weaker, but the rate of the convergence is pleasant in this case. The idea of the proof follows the procedure inspired by [2], in which the authors consider a relative simple framework (without the nonlinear term and  $f$  is bounded). In our case, it is quite non-trivial to deal with the nonlinear term and unbounded  $f$ .

The most challenge in studying the strong convergence is how to deal with the nonlinear term in Burger's equation. To overcome this difficulty, we first give some estimates of the slow component  $X_t^\varepsilon$  and fast component  $Y_t^\varepsilon$  in  $H$ . Secondly, by using the smoothness of semigroup  $e^{t\Delta}$  and interpolation inequality, we can further obtain  $\sup_{\varepsilon \in (0,1)} \mathbb{E} \sup_{t \in [0,T]} |X_t^\varepsilon|_\alpha^p \leq C_{p,T}$ , which is a key step for proving the (1.3), where  $|\cdot|_\alpha$  is Sobolev norm. Finally, applying the skill of stopping time and following the procedure inspired by [12], we can obtain the main result.

Finally, we refer that, in recent years, there are many interesting results for stochastic Burger's equation [5, 6, 7, 8, 9, 10, 16, 22].

The paper is organized as follows. In the next section, we introduce some notation and assumptions that we use throughout the paper. In section 3, we give out our main results. Section 4 and Section 5 are devoted to prove the strong convergence and weak convergence respectively.

Along the paper,  $C$ ,  $C_p$  and  $C_{p,T}$  will denote positive constants which may change from line to line, where  $C_p$  depends on  $p$ ,  $C_{p,T}$  depends on  $p, T$ .

## 2 Preliminaries

Denote by  $|\cdot|_{L^p}$  the usual norm of the space  $L^p(0,1)$ ,  $p \geq 1$ , and by  $|\cdot|_\infty$  the usual supremum norm of  $L^\infty(0,1)$ . We consider the separable Hilbert space  $H = L^2(0,1)$  (the inner product denoted  $\langle \cdot, \cdot \rangle$ ). As usual, for  $k \in \mathbb{N}$ ,  $p \geq 1$ ,  $W^{k,p}(0,1)$  is the Sobolev space of all functions in  $L^p(0,1)$  whose differentials belong to  $L^p(0,1)$  up to the order  $k$ . Recall that the usual Sobolev space  $W^{k,p}(0,1)$  can be extended to the  $W^{s,p}(0,1)$ , for  $s \in \mathbb{R}$ . Set  $H^k(0,1) \triangleq W^{k,2}(0,1)$  and  $H_0^1(0,1)$  is the subspace of  $H^1(0,1)$  of all functions whose trace at 0 and 1 vanishes. We define the unbounded self-adjoint operator  $A$  by

$$Ax = \Delta x = \frac{\partial^2}{\partial \xi^2} x, \quad x \in \mathcal{D}(A) = H^2(0,1) \cap H_0^1(0,1).$$

Note that the operator  $A$  is the infinitesimal generator of a strongly continuous semigroup in  $H$ , which we denote by  $e^{tA}$ ,  $t \geq 0$ . It is well known that  $e^{tA}$  ( $t \geq 0$ ) have smoothing properties, that is, for any  $s_1, s_2 \in \mathbb{R}$  with  $s_1 \leq s_2$ ,  $r \geq 1$ ,  $e^{tA} : W^{s_1,r}(0,1) \rightarrow W^{s_2,r}(0,1)$  and there exists a constant  $C$  which depending on  $s_1, s_2, r$  such that

$$|e^{tA} z|_{W^{s_2,r}(0,1)} \leq C (1 + t^{(s_1-s_2)/2}) |z|_{W^{s_1,r}(0,1)}, \quad z \in W^{s_1,r}(0,1). \quad (2.5)$$

$(-A)^\alpha$  is the power of the operator  $-A$ , and  $|\cdot|_\alpha$  is the norm of  $\mathcal{D}((-A)^\alpha)$  which is equivalent to the norm of  $H^\alpha(0,1) \triangleq W^{\alpha,2}(0,1)$ . We have  $|\cdot|_0 = |\cdot|_{L^2}$ , and denote it by  $|\cdot|$  for simplicity. Then,

$$e_k(\xi) = \sqrt{2} \sin(k\pi\xi), \quad \xi \in [0,1], k \in \mathbb{N}$$

are eigenfunctions of  $-A$  with eigenvalue  $\lambda_k = k^2\pi^2$ .

Define the bilinear operator

$$B(x, y) : H \times H_0^1(0, 1) \rightarrow H_0^{-1}(0, 1), \quad B(x, y) = x \cdot \partial_\xi y,$$

and the trilinear operator

$$b(x, y, z) : H \times H_0^1(0, 1) \times H \rightarrow \mathbb{R}, \quad b(x, y, z) = \int_0^1 x(\xi) \partial_\xi y(\xi) z(\xi) d\xi.$$

It is convenient to put  $B(x) = B(x, x)$ , for  $x \in H_0^1(0, 1)$ .

The following several properties of  $b(\cdot, \cdot, \cdot)$  and  $B(\cdot, \cdot)$  are well-known (for example see [9]) and will be used later on.

**Lemma 2.1** *For any  $x, y \in H_0^1(0, 1)$ ,*

$$b(x, x, y) = -\frac{1}{2}b(x, y, x), \quad b(x, y, y) = 0.$$

□

**Lemma 2.2** *Suppose  $\alpha_i \geq 0$  ( $i = 1, 2, 3$ ) satisfies one of the following conditions*

(1)  $\alpha_i \neq \frac{1}{2}$  ( $i = 1, 2, 3$ ),  $\alpha_1 + \alpha_2 + \alpha_3 \geq \frac{1}{2}$ ,

(2)  $\alpha_i = \frac{1}{2}$  for some  $i$ ,  $\alpha_1 + \alpha_2 + \alpha_3 > \frac{1}{2}$ ,

*then  $b$  is continuous from  $H^{\alpha_1}(0, 1) \times H^{\alpha_2+1}(0, 1) \times H^{\alpha_3}(0, 1)$  to  $\mathbb{R}$ , i.e.*

$$|b(x, y, z)| \leq C|x|_{\alpha_1}|y|_{\alpha_2+1}|z|_{\alpha_3}.$$

□

The following inequalities can be derived by the above lemma.

**Corollary 2.1** *For any  $x \in H_0^1(0, 1)$ , we have*

(1)  $|B(x)| \leq C|x|_1^2$ .

(2)  $|B(x)|_{-1} \leq C|x| \cdot |x|_1$ .

□

**Lemma 2.3** *For any  $x, y \in H_0^1(0, 1)$ , we have*

(1)  $|B(x) - B(y)| \leq C|x - y|_1(|x|_1 + |y|_1)$ .

(2)  $|B(x) - B(y)|_{-1} \leq C|x - y|(|x|_1 + |y|_1)$ .

□

With the notations we have introduced, system (1.1) can be rewritten as the following form:

$$\begin{cases} dX_t^\varepsilon = [AX_t^\varepsilon + B(X_t^\varepsilon) + f(X_t^\varepsilon, Y_t^\varepsilon)]dt + dW_t^{Q_1}, & X_0^\varepsilon = x \\ dY_t^\varepsilon = \frac{1}{\varepsilon}[AY_t^\varepsilon + g(X_t^\varepsilon, Y_t^\varepsilon)]dt + \frac{1}{\sqrt{\varepsilon}}dW_t^{Q_2}, & Y_0^\varepsilon = y \\ X_t^\varepsilon(0) = X_t^\varepsilon(1) = Y_t^\varepsilon(0) = Y_t^\varepsilon(1) = 0, \end{cases} \quad (2.6)$$

where  $Q_1$ -Wiener process  $W_t^{Q_1}$  is given by

$$W_t^{Q_1} = \sum_{k=1}^{\infty} \sqrt{\alpha_k} \beta_t^k e_k, \quad t \geq 0,$$

where  $\alpha_k \geq 0$  satisfies  $Q_1 e_k = \alpha_k e_k$  with  $\sum_{k=1}^{\infty} \alpha_k < +\infty$ , and  $\{\beta^k\}_{k \in \mathbb{N}}$  is a sequence of mutually independent standard Brownian motions. We also assume  $\text{Tr} Q_2 < \infty$ .

Now we make the following basic assumptions on the coefficients  $f$  and  $g$  throughout this paper.

**(H1)** The functions  $f, g : H \times H \rightarrow H$  satisfy the global Lipschitz condition, i.e., there exist positive constants  $L_f$  and  $L_g$  such that for any  $x_1, x_2, y_1, y_2 \in H$ ,

$$|f(x_1, y_1) - f(x_2, y_2)| \leq L_f(|x_1 - x_2| + |y_1 - y_2|) \quad (2.7)$$

and

$$|g(x_1, y_1) - g(x_2, y_2)| \leq L_g(|x_1 - x_2| + |y_1 - y_2|). \quad (2.8)$$

**(H2)** The growth rate of nonlinear term  $g$  in fast component equation is smaller than the decay rate of operator  $\Delta$ , i.e.,

$$\eta := \lambda_1 - L_g > 0. \quad (2.9)$$

Refer to [7], we have

**Theorem 2.1** *Suppose that the condition **(H1)** holds. For any given initial value  $x, y \in H$ , there exists a unique mild solution  $\{(X_t^\varepsilon, Y_t^\varepsilon), t \geq 0\}$  to system (2.6) and for all  $T > 0$ ,  $(X^\varepsilon, Y^\varepsilon) \in C([0, T]; H) \times C([0, T]; H)$ ,  $\mathbb{P}$ -a.s..*

$$\begin{cases} X_t^\varepsilon = e^{tA}x + \int_0^t e^{(t-s)A}B(X_s^\varepsilon)ds + \int_0^t e^{(t-s)A}f(X_s^\varepsilon, Y_s^\varepsilon)ds + \int_0^t e^{(t-s)A}dW_s^{Q_1}, \\ Y_t^\varepsilon = e^{tA/\varepsilon}y + \frac{1}{\varepsilon} \int_0^t e^{(t-s)A/\varepsilon}g(X_s^\varepsilon, Y_s^\varepsilon)ds + \frac{1}{\sqrt{\varepsilon}} \int_0^t e^{(t-s)A/\varepsilon}dW_s^{Q_2}. \end{cases} \quad (2.10)$$

At the end of this section, we show the classical Gronwall inequality and a Gronwall-type inequality, which will be used later.

**Lemma 2.4** (*Gronwall inequality*)

Let  $\alpha, \beta$  be real-value functions defined on  $[0, T]$ , assume that  $\beta$  and  $u$  are continuous and that the negative part of  $\alpha$  is integrable on every closed and bounded subinterval of  $[0, T]$ .

(a) If  $\beta$  is non-negative and if  $u$  satisfies the integral inequality

$$u_t \leq \alpha_t + \int_0^t \beta_s u_s ds, \quad \forall t \in [0, T],$$

then

$$u_t \leq \alpha_t + \int_0^t \alpha_s \beta_s \exp\left(\int_s^t \beta_r dr\right) ds, \quad \forall t \in [0, T].$$

(b) If, in addition, the function  $\alpha$  is nondecreasing, then

$$u_t \leq \alpha_t \exp\left(\int_0^t \beta_r dr\right), \quad \forall t \in [0, T].$$

**Lemma 2.5** (*Gronwall-type inequality*)

For any given  $\alpha \in (0, 1)$ ,  $\beta \in (0, 1)$ , if  $f(t)$  is a non-negative real-valued integrable function on  $[0, T]$ , and the following inequality holds for some positive constants  $C_1, C_2$

$$f(t) \leq C_1 t^{-\alpha} + C_2 \int_0^t (t-s)^{-\beta} f(s) ds, \quad \forall t \in [0, T],$$

then there exists some  $k \in \mathbb{N}$  satisfying

$$f(t) \leq CC_1 t^{-\alpha} e^{CC_2^k}, \quad \forall t \in [0, T],$$

where  $C$  is a positive constant depending on  $\alpha, \beta$  and  $T$ .

*Proof* By iterating and Fubini theorem, we have

$$\begin{aligned}
f(t) &\leq C_1 t^{-\alpha} + C_2 \int_0^t (t-s)^{-\beta} \left[ C_1 s^{-\alpha} + C_2 \int_0^s (s-r)^{-\beta} f(r) dr \right] ds \\
&\leq C_1 t^{-\alpha} + C_1 C_2 \int_0^t (t-s)^{-\beta} s^{-\alpha} ds + C_2^2 \int_0^t f(r) \left[ \int_r^t (t-s)^{-\beta} (s-r)^{-\beta} ds \right] dr \\
&\leq C C_1 (C_2 + 1) t^{-\alpha} + C C_2^2 \int_0^t (t-r)^{1-2\beta} f(r) dr,
\end{aligned}$$

where  $C$  is a constant depending on  $\alpha, \beta, T$ .

If  $1 - 2\beta \geq 0$ , then we can easily obtain the result by Lemma 2.4. However, if  $1 - 2\beta < 0$ , then after iterating finite times, there exist  $\gamma \geq 0, k_1, k_2 \in \mathbb{N}$  such that

$$\begin{aligned}
f(t) &\leq C C_1 (C_2^{k_1} + 1) t^{-\alpha} + C C_2^{k_2} \int_0^t (t-r)^\gamma f(r) dr \\
&\leq C C_1 (C_2^{k_1} + 1) t^{-\alpha} + C C_2^{k_2} \int_0^t f(r) dr.
\end{aligned}$$

Finally by Lemma 2.4, we obtain

$$\begin{aligned}
f(t) &\leq C C_1 (C_2^{k_1} + 1) t^{-\alpha} e^{C C_2^{k_2}} \\
&\leq C C_1 t^{-\alpha} e^{C C_2^k},
\end{aligned}$$

for some  $k \in \mathbb{N}$ . □

### 3 Main results

In this paper, we mainly study the speed of  $X^\varepsilon$  convergent to  $\bar{X}$  in the strong and weak sense respectively. We give the main results in this section and provide the proofs in the next two sections.

More conditions are needed to study the strong and weak convergence, which are stated as follows:

**(H3)** There exist constants  $\alpha \in (1, \frac{3}{2})$  and  $\beta \in (0, \frac{1}{2})$  such that

$$\sum_{k=1}^{\infty} \alpha_k \lambda_k^{\alpha+2\beta-1} < +\infty. \tag{3.1}$$

**(H4)** Assume  $f$  and  $g$  are twice differentiable with respect to first and second variable respectively, and following inequalities hold:

(1) For any  $x, y \in H$  and  $h, k \in H$ ,

$$|D_{xx}^2 f(x, y)(h, k)| \leq C|h| \cdot |k|.$$

(2) For any  $x, y \in H$  and  $h, k \in H$ ,

$$|D_y g(x, y) \cdot h| \leq C|h|, \quad |D_{yy}^2 g(x, y)(h, k)| \leq C|h| \cdot |k|.$$

(3) For any  $x, y \in H$ , there exists a positive constant  $C$  such that

$$|\langle f(x, y), x \rangle| \leq C(1 + |x|^2).$$

Now, our main results are following:

**Theorem 3.1 (Strong convergence)** Assume **(H1)**, **(H2)** and **(H3)** hold, then for any  $x \in H^\alpha, y \in H, p, T > 0$ , we have

$$\mathbb{E} \sup_{0 \leq t \leq T} |X_t^\varepsilon - \bar{X}_t|^{2p} \leq C \left( \frac{1}{-\log \varepsilon} \right)^{\frac{1}{4p}} \rightarrow 0 \quad (\varepsilon \rightarrow 0),$$

where  $C$  is a positive constant which only depends on  $T, p, |x|_\alpha$  and  $|y|$ .

**Theorem 3.2 (Weak convergence 1)** Assume **(H1)**, **(H2)** and **(H4)** hold, and  $Q_1 = 0$ , then for any  $\theta \in (0, 1], r \in (0, 1), x \in H^\theta, y \in H, \phi \in C_b^2, t \in (0, T], \delta \in (0, \frac{1}{2})$ , and for any small enough  $\varepsilon \in (0, 1)$ , we have

$$|\mathbb{E}[\phi(X_t^\varepsilon)] - \mathbb{E}[\phi(\bar{X}_t)]| \leq C(1 + t^{-\theta + \frac{\theta^2}{1+\delta}})\varepsilon^{1-r},$$

where  $C$  is a positive constant which only depends on  $T, |x|_\theta, |y|, \delta$  and  $\phi$ .

**Theorem 3.3 (Weak convergence 2)** Under the same assumptions in Theorem 3.2 with  $\theta \in (1, \frac{3}{2})$ , then for any  $t \in (0, T], r \in (0, 1)$ , we can obtain

$$|\mathbb{E}[\phi(X^\varepsilon(t))] - \mathbb{E}[\phi(\bar{X}(t))]| \leq C\varepsilon^{1-r},$$

where  $C$  is a positive constant which only depends on  $T, |x|_\theta, |y|$  and  $\phi$ .

**Remark 3.1** Theorem 3.3 shows that if the initial value  $x$  has higher regularity, then the control constant becomes quite terse.

**Remark 3.2** We assume (3) of **(H4)** holds and  $Q_1 = 0$  in studying the weak convergence for several technical difficulty, for example, see Lemma 5.8, that for any  $x \in H$  there exists a positive constant  $C$  such that  $\sup_{t \in [0, T]} |X^\varepsilon(t)| \leq C(1 + |x|)$ , and then  $\sup_{t \in [0, T]} e^{|X^\varepsilon(t)|^k} \leq C$ , which plays an important role in proving Theorem 3.2 and Theorem 3.3, see Subsection 5.5. For the case of  $Q_1 \neq 0$ , these estimates are failed and we do not know how to deal with this case recently.

## 4 Strong convergence

In this section, we intend to study the strong convergence. We first give some estimates of the solution  $(X_t^\varepsilon, Y_t^\varepsilon)$ . Secondly, following the idea inspired by Khasminskii in [17], we introduce an auxiliary process  $(\hat{X}_t^\varepsilon, \hat{Y}_t^\varepsilon) \in H \times H$  and also give their estimates. Meanwhile, we show the error of  $X_t^\varepsilon - \hat{X}_t^\varepsilon$ . Finally, we study the average equation and apply the skill of stopping time and following the procedure inspired by [12], the error of  $\hat{X}_t^\varepsilon - \bar{X}_t^\varepsilon$  is obtained. Hence, the strong convergence is easily proved. Notice that we always assume condition **(H3)** holds in this section.

### 4.1 Some priori estimates of $(X_t^\varepsilon, Y_t^\varepsilon)$

At first, we prove uniform bounds with respect to  $\varepsilon \in (0, 1)$  for  $p$ -moment of the solutions  $X_t^\varepsilon$  and  $Y_t^\varepsilon$  for the system (2.6).

**Lemma 4.1** For any  $x, y \in H$ ,  $p \geq 2$  and  $T > 0$ , there exists a constant  $C_{p,T} > 0$  such that

$$\sup_{\varepsilon \in (0,1)} \sup_{0 \leq t \leq T} \mathbb{E}|X_t^\varepsilon|^{2p} \leq C_{p,T}(1 + |x|^{2p} + |y|^{2p}) \quad (4.1)$$

and

$$\sup_{\varepsilon \in (0,1)} \sup_{0 \leq t \leq T} \mathbb{E}|Y_t^\varepsilon|^{2p} \leq C_{p,T}(1 + |x|^{2p} + |y|^{2p}). \quad (4.2)$$

*Proof* According to Itô formula

$$\begin{aligned} \mathbb{E}|Y_t^\varepsilon|^{2p} &= |y|^{2p} + \frac{2p}{\varepsilon} \mathbb{E} \int_0^t |Y_s^\varepsilon|^{2p-2} \langle AY_s^\varepsilon, Y_s^\varepsilon \rangle ds + \frac{2p}{\varepsilon} \mathbb{E} \int_0^t |Y_s^\varepsilon|^{2p-2} \langle g(X_s^\varepsilon, Y_s^\varepsilon), Y_s^\varepsilon \rangle ds \\ &\quad + \frac{p}{\varepsilon} \mathbb{E} \int_0^t |Y_s^\varepsilon|^{2p-2} \text{Tr} Q_2 ds + \frac{2p(p-1)}{\varepsilon} \mathbb{E} \int_0^t |Y_s^\varepsilon|^{2p-4} |\sqrt{Q_2} Y_s^\varepsilon|^2 ds. \end{aligned}$$

Then, there exists a constant  $\gamma > 0$  such that

$$\begin{aligned} \frac{d}{dt} \mathbb{E}|Y_t^\varepsilon|^{2p} &= \frac{2p}{\varepsilon} \mathbb{E} \left( |Y_t^\varepsilon|^{2p-2} \langle AY_t^\varepsilon, Y_t^\varepsilon \rangle \right) + \frac{2p}{\varepsilon} \mathbb{E} \left[ |Y_t^\varepsilon|^{2p-2} \langle g(X_t^\varepsilon, Y_t^\varepsilon), Y_t^\varepsilon \rangle \right] \\ &\quad + \frac{p}{\varepsilon} \mathbb{E}|Y_t^\varepsilon|^{2p-2} \text{Tr} Q_2 + \frac{2p(p-1)}{\varepsilon} \mathbb{E} \left\{ |Y_t^\varepsilon|^{2p-4} |\sqrt{Q_2} Y_t^\varepsilon|^2 \right\} \\ &\leq -\frac{2p\lambda_1}{\varepsilon} \mathbb{E}|Y_t^\varepsilon|^{2p} + \frac{2p}{\varepsilon} \mathbb{E} \left\{ |Y_t^\varepsilon|^{2p-2} [C|Y_t^\varepsilon| + L_g(|X_t^\varepsilon| \cdot |Y_t^\varepsilon| + |Y_t^\varepsilon|^2)] \right\} \\ &\quad + \frac{p}{\varepsilon} \mathbb{E}|Y_t^\varepsilon|^{2p-2} \text{Tr} Q_2 + \frac{2p(p-1)}{\varepsilon} \mathbb{E}|Y_t^\varepsilon|^{2p-2} \text{Tr} Q_2 \\ &\leq -\frac{p\gamma}{\varepsilon} \mathbb{E}|Y_t^\varepsilon|^{2p} + \frac{C_p}{\varepsilon} \mathbb{E}|X_t^\varepsilon|^{2p} + \frac{C_p}{\varepsilon}, \end{aligned} \quad (4.3)$$

where the last inequality by the fact of  $\lambda_1 - L_g > 0$  in **(H2)** and Young inequality. Hence, by comparison theorem

$$\mathbb{E}|Y_t^\varepsilon|^{2p} \leq |y|^{2p} e^{-\frac{p\gamma}{\varepsilon}t} + \frac{C_p}{\varepsilon} \int_0^t e^{-\frac{p\gamma}{\varepsilon}(t-s)} \left( 1 + \mathbb{E}|X_s^\varepsilon|^{2p} \right) ds. \quad (4.4)$$

By Itô formula again

$$\begin{aligned} |X_t^\varepsilon|^{2p} &= |x|^{2p} + 2p \int_0^t |X_s^\varepsilon|^{2p-2} \langle AX_s^\varepsilon, X_s^\varepsilon \rangle ds + 2p \int_0^t |X_s^\varepsilon|^{2p-2} \langle B(X_s^\varepsilon), X_s^\varepsilon \rangle ds \\ &\quad + 2p \int_0^t |X_s^\varepsilon|^{2p-2} \langle f(X_s^\varepsilon, Y_s^\varepsilon), X_s^\varepsilon \rangle ds + 2p \int_0^t |X_s^\varepsilon|^{2p-2} \langle X_s^\varepsilon, dW_s^{Q_1} \rangle \\ &\quad + p \int_0^t |X_s^\varepsilon|^{2p-2} \text{Tr} Q_1 ds + 2p(p-1) \int_0^t |X_s^\varepsilon|^{2p-4} |\sqrt{Q_1} X_s^\varepsilon|^2 ds. \end{aligned}$$

Then, similar as we did in (4.3), we have

$$\begin{aligned} \frac{d}{dt} \mathbb{E}|X_t^\varepsilon|^{2p} &= 2p \mathbb{E} \left( |X_t^\varepsilon|^{2p-2} \langle AX_t^\varepsilon, X_t^\varepsilon \rangle \right) + 2p \mathbb{E} \left[ |X_t^\varepsilon|^{2p-2} \langle f(X_t^\varepsilon, Y_t^\varepsilon), X_t^\varepsilon \rangle \right] \\ &\quad + p \mathbb{E}|X_t^\varepsilon|^{2p-2} \text{Tr} Q_1 + 2p(p-1) \mathbb{E} \left\{ |X_t^\varepsilon|^{2p-4} |\sqrt{Q_1} X_t^\varepsilon|^2 \right\} \\ &\leq C_p \mathbb{E}|X_t^\varepsilon|^{2p} + C_p \mathbb{E}|Y_t^\varepsilon|^{2p} + C_p. \end{aligned}$$



Hence, by comparison theorem

$$\mathbb{E}|X_t^\varepsilon|^{2p} \leq |x|^{2p} e^{C_p t} + C_p \int_0^t e^{C_p(t-s)} (1 + \mathbb{E}|Y_s^\varepsilon|^{2p}) ds. \quad (4.5)$$

Combining (4.4) and (4.5), for any  $t \leq T$

$$\begin{aligned} \mathbb{E}|Y_t^\varepsilon|^{2p} &\leq C_{p,T}(1 + |x|^{2p} + |y|^{2p}) + \frac{C_p}{\varepsilon} \int_0^t e^{-\frac{p\gamma}{\varepsilon}(t-s)} \int_0^s \mathbb{E}|Y_r^\varepsilon|^{2p} dr ds \\ &\quad + \frac{C_p}{\varepsilon} \int_0^t e^{-\frac{p\gamma}{\varepsilon}(t-s)} ds. \end{aligned}$$

With a change of variable, we have

$$\begin{aligned} \mathbb{E}|Y_t^\varepsilon|^{2p} &\leq C_{p,T}(1 + |x|^{2p} + |y|^{2p}) + C_p \int_0^t \left[ \int_0^{\frac{t-r}{\varepsilon}} e^{-p\gamma s} ds \right] \mathbb{E}|Y_r^\varepsilon|^{2p} dr \\ &\quad + C_p \int_0^{t/\varepsilon} e^{-p\gamma s} ds. \\ &\leq C_{p,T}(1 + |x|^{2p} + |y|^{2p}) + C_p \int_0^t \mathbb{E}|Y_r^\varepsilon|^{2p} dr. \end{aligned}$$

The Grownall's inequality implies

$$\mathbb{E}|Y_t^\varepsilon|^{2p} \leq C_{p,T}(1 + |x|^{2p} + |y|^{2p}),$$

which also gives

$$\mathbb{E}|X_t^\varepsilon|^{2p} \leq C_{p,T}(1 + |x|^{2p} + |y|^{2p}).$$

The proof is complete. □

In order to estimate the high-order norm of  $X^\varepsilon$ , we recall the stochastic convolution

$$W_A(t) := \int_0^t e^{(t-s)A} dW_s^{Q_1}.$$

Then we have the following result:

**Lemma 4.2** *For any  $p, T > 0$ , we have*

$$\mathbb{E} \sup_{0 \leq t \leq T} |W_A(t)|_\alpha^{2p} \leq C_{p,T}, \quad (4.6)$$

where  $\alpha$  is the one in **(H3)**.

*Proof* By Hölder inequality, it is suffice to prove (4.6) holds when  $p$  is large enough. Using the factorization method, we can write

$$W_A(t) = \frac{\sin \pi \beta}{\pi} \int_0^t e^{(t-s)A} (t-s)^{\beta-1} Z_s ds,$$

where  $\beta$  is the one in **(H3)** and

$$Z_s = \int_0^s e^{(s-r)A} (s-r)^{-\beta} dW_r^{Q_1}.$$

For any  $T > 0$ ,  $t \in [0, T]$ ,  $p$  is large enough such that  $\frac{2p(1-\beta)}{2p-1} < 1$ , we have

$$\begin{aligned} |W_A(t)|_\alpha &\leq C \left( \int_0^t (t-s)^{-\frac{2p(1-\beta)}{2p-1}} ds \right)^{\frac{2p-1}{2p}} |Z|_{L^{2p}(0,T;H^\alpha)} \\ &\leq C_p t^{\beta-\frac{1}{2p}} |Z|_{L^{2p}(0,T;H^\alpha)}. \end{aligned}$$

Then

$$\sup_{t \leq T} |W_A(t)|_\alpha^{2p} \leq C_{p,T} |Z|_{L^{2p}(0,T;H^\alpha)}^{2p}. \quad (4.7)$$

Notice that  $(-A)^{\alpha/2} Z_s \sim N(0, \tilde{Q}_s)$  which is a Gaussian random variable with mean zero and covariance operator

$$\tilde{Q}_s x = \int_0^s r^{-2\beta} e^{rA} (-A)^\alpha Q_1 e^{rA^*} x dr.$$

Then for any  $p \geq 1$ ,  $s > 0$ , by [6, Corollary 2.17]

$$\begin{aligned} \mathbb{E} |(-A)^{\alpha/2} Z_s|^{2p} &\leq C_p [\text{Tr}(\tilde{Q}_s)]^p \\ &= C_p \left( \sum_k \int_0^s r^{-2\beta} e^{-2r\lambda_k} \lambda_k^\alpha \alpha_k dr \right)^p \\ &= C_p \left( \sum_k \lambda_k^{\alpha+2\beta-1} \alpha_k \int_0^{2s\lambda_k} r^{-2\beta} e^{-r} dr \right)^p \\ &\leq C_p (\sum_k \lambda_k^{\alpha+2\beta-1} \alpha_k)^p, \end{aligned} \quad (4.8)$$

where the last inequality by the fact of

$$\int_0^{2s\lambda_k} r^{-2\beta} e^{-r} dr \leq \int_0^\infty r^{-2\beta} e^{-r} dr < \infty.$$

Hence, (4.7) and (4.8) imply

$$\mathbb{E} \sup_{0 \leq t \leq T} |W_A(t)|_\alpha^{2p} \leq C_{p,T} \int_0^T \mathbb{E} |Z_s|_\alpha^{2p} ds \leq C_{p,T}.$$

□

**Lemma 4.3** *For any  $x \in H^\alpha$ ,  $y \in H$ ,  $T > 0$  and  $p > 0$ , there exists a positive constant  $C_{p,T}$  independent of  $\varepsilon$ , such that*

$$\mathbb{E} \left( \sup_{t \in [0,T]} |X_t^\varepsilon|_\alpha^{2p} \right) \leq C_{p,T} (|x|_\alpha^{2p} + |y|^{2p} + 1), \quad (4.9)$$

where  $\alpha$  is the one in **(H3)**.

*Proof* It is also suffice to prove (4.9) holds when  $p$  is large enough. Recall that

$$X_t^\varepsilon = e^{tA} x + \int_0^t e^{(t-s)A} B(X_s^\varepsilon) ds + \int_0^t e^{(t-s)A} f(X_s^\varepsilon, Y_s^\varepsilon) ds + \int_0^t e^{(t-s)A} dW_s^{Q_1}.$$

For the first term, it is well known that  $|e^{At} x|_\alpha^{2p} \leq |x|_\alpha^{2p}$ . For the second term, according to (2.5) and Lemma 2.2, we have the following estimate

$$\left| \int_0^t e^{(t-s)A} B(X_s^\varepsilon) ds \right|_\alpha \leq \int_0^t |e^{(t-s)A} B(X_s^\varepsilon)|_\alpha ds$$

$$\begin{aligned}
&\leq C \int_0^t (1 + (t-s)^{\frac{-\alpha_3-\alpha}{2}}) |B(X_s^\varepsilon)|_{-\alpha_3} ds \\
&\leq C \int_0^t (1 + (t-s)^{\frac{-\alpha_3-\alpha}{2}}) |X_s^\varepsilon|_{\alpha_1} |X_s^\varepsilon|_{\alpha_2+1} ds,
\end{aligned}$$

where  $\alpha_1 + \alpha_2 + \alpha_3 > \frac{1}{2}$  and  $\alpha_i > 0, i = 1, 2, 3$ . Thanks to interpolation inequality,

$$|X_s^\varepsilon|_{\alpha_1} \leq C |X_s^\varepsilon|^{\frac{\alpha-\alpha_1}{\alpha}} |X_s^\varepsilon|^{\frac{\alpha_1}{\alpha}} \quad (4.10)$$

holds for any  $0 < \alpha_1 < \alpha$ , and

$$|X_s^\varepsilon|_{\alpha_2+1} \leq C |X_s^\varepsilon|^{\frac{\alpha-\alpha_2-1}{\alpha}} |X_s^\varepsilon|^{\frac{\alpha_2+1}{\alpha}} \quad (4.11)$$

holds for any  $0 < \alpha_2 + 1 < \alpha$ . Besides, we also assume  $0 < 1 + \alpha_1 + \alpha_2 < \alpha$ . Then, combining (4.10) and (4.11), we obtain

$$\begin{aligned}
&\mathbb{E} \sup_{t \in [0, T]} \left| \int_0^t e^{(t-s)A} B(X_s^\varepsilon) ds \right|_\alpha^{2p} \\
&\leq C \mathbb{E} \left( \sup_{t \in [0, T]} \int_0^t (1 + (t-s)^{\frac{-\alpha_3-\alpha}{2}}) |X_s^\varepsilon|^{\frac{2\alpha-\alpha_1-\alpha_2-1}{\alpha}} |X_s^\varepsilon|^{\frac{\alpha_1+\alpha_2+1}{\alpha}} ds \right)^{2p} \\
&\leq C_{p,T} \mathbb{E} \left[ \left( \sup_{t \in [0, T]} \int_0^t (1 + (t-s)^{\frac{-\alpha_3-\alpha}{2}})^{\frac{2p}{2p-1}} ds \right)^{2p-1} \left( \sup_{t \in [0, T]} \int_0^t |X_s^\varepsilon|^{2p \cdot \frac{2\alpha-\alpha_1-\alpha_2-1}{\alpha}} |X_s^\varepsilon|^{2p \cdot \frac{\alpha_1+\alpha_2+1}{\alpha}} ds \right) \right] \\
&\leq C_{p,T} \left( 1 + \int_0^T s^{-\frac{\alpha_3+\alpha}{2} \cdot \frac{2p}{2p-1}} ds \right)^{2p-1} \left( \int_0^T \mathbb{E} |X_s^\varepsilon|^{2p \cdot \frac{2\alpha-\alpha_1-\alpha_2-1}{\alpha}} ds + \int_0^T \mathbb{E} |X_s^\varepsilon|_\alpha^{2p} ds \right).
\end{aligned}$$

By Lemma 4.1, we obtain

$$\begin{aligned}
&\mathbb{E} \sup_{t \in [0, T]} \left| \int_0^t e^{(t-s)A} B(X_s^\varepsilon) ds \right|_\alpha^{2p} \\
&\leq C_{p,T} \left( 1 + \int_0^T s^{-\frac{\alpha_3+\alpha}{2} \cdot \frac{2p}{2p-1}} ds \right)^{2p-1} \left( 1 + \int_0^T \mathbb{E} |X_s^\varepsilon|_\alpha^{2p} ds \right).
\end{aligned}$$

Now, taking  $p$  large enough such that

$$\frac{\alpha + \alpha_3}{2} \cdot \frac{2p}{2p-1} < 1$$

and positive constants  $\alpha_1, \alpha_2, \alpha_3$  such that

$$0 < 1 + \alpha_1 + \alpha_2 < \alpha, \quad \alpha_1 + \alpha_2 + \alpha_3 > \frac{1}{2}$$

hold at the same time. For instance, we can take  $\alpha_3 = \frac{1}{2}$ ,  $\alpha_1 = \alpha_2 = \frac{\alpha-1}{4}$ , and  $p > \frac{2}{3-2\alpha}$ . Hence, we can get

$$\mathbb{E} \left( \sup_{t \in [0, T]} \left| \int_0^t e^{(t-s)A} B(X_s^\varepsilon) ds \right|_\alpha^{2p} \right) \leq C_{p,T} \left( 1 + \int_0^T \mathbb{E} |X_s^\varepsilon|_\alpha^{2p} ds \right). \quad (4.12)$$

For the third term, according to (2.5),

$$\mathbb{E} \left( \sup_{t \in [0, T]} \left| \int_0^t e^{(t-s)A} f(X_s^\varepsilon, Y_s^\varepsilon) ds \right|_\alpha^{2p} \right)$$

$$\begin{aligned}
&\leq C\mathbb{E}\left[\sup_{t\in[0,T]}\int_0^t(1+(t-s)^{-\frac{\alpha}{2}})|f(X_s^\varepsilon, Y_s^\varepsilon)|ds\right]^{2p} \\
&\leq C\mathbb{E}\left[\sup_{t\in[0,T]}\int_0^t(1+(t-s)^{-\frac{\alpha}{2}})(1+|X_s^\varepsilon|+|Y_s^\varepsilon|)ds\right]^{2p} \\
&\leq C_p\left(\sup_{t\in[0,T]}\int_0^t(1+(t-s)^{-\frac{\alpha}{2}})^{\frac{2p}{2p-1}}ds\right)^{2p-1}\mathbb{E}\left(\sup_{t\in[0,T]}\int_0^t(1+|X_s^\varepsilon|+|Y_s^\varepsilon|)^{2p}ds\right) \\
&\leq C_{p,T}\left(1+\int_0^T s^{-\frac{\alpha p}{2p-1}}ds\right)^{2p-1}\int_0^T(1+\mathbb{E}|X_s^\varepsilon|^{2p}+\mathbb{E}|Y_s^\varepsilon|^{2p})ds.
\end{aligned}$$

Taking  $p$  large enough such that  $\frac{\alpha p}{2p-1} < 1$  and using Lemma 4.1, we get

$$\mathbb{E}\left(\sup_{t\in[0,T]}\left|\int_0^t e^{(t-s)A}f(X_s^\varepsilon, Y_s^\varepsilon)ds\right|_\alpha^{2p}\right) \leq C_{p,T}(1+|x|_\alpha^{2p}+|y|_\alpha^{2p}). \quad (4.13)$$

Therefore, by (4.12), (4.13) and Lemma 4.2, applying Gronwall's inequality, we finally have

$$\sup_{\varepsilon\in(0,1)}\mathbb{E}\left(\sup_{t\in[0,T]}|X_t^\varepsilon|_\alpha^{2p}\right) \leq C_{p,T}(1+|x|_\alpha^{2p}+|y|_\alpha^{2p}),$$

which completes the proof.  $\square$

Next, we are going to provide a Hölder continuity of time variable for  $X_t^\varepsilon$ .

**Lemma 4.4** *For any  $x \in H^\alpha, y \in H, T > 0, 0 < t \leq t+h \leq T$ , then there exists a positive constant  $C_{p,T}$  such that*

$$\sup_{\varepsilon\in(0,1)}\mathbb{E}|X_{t+h}^\varepsilon - X_t^\varepsilon|^{2p} \leq C_{p,T}h^p(1+|x|_\alpha^{2p}+|y|_\alpha^{2p}).$$

*Proof* After simple calculations, we have

$$\begin{aligned}
X_{t+h}^\varepsilon - X_t^\varepsilon &= (e^{Ah} - I)X_t^\varepsilon + \int_t^{t+h} e^{(t+h-s)A}B(X_s^\varepsilon)ds \\
&\quad + \int_t^{t+h} e^{(t+h-s)A}f(X_s^\varepsilon, Y_s^\varepsilon)ds + \int_t^{t+h} e^{(t+h-s)A}dW_s^{Q_1} \\
&:= I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

For  $\alpha$  given in **(H3)**, there exists a constant  $C_\alpha > 0$  such that for all  $x \in \mathcal{D}((-A)^{\frac{\alpha}{2}})$ ,

$$|e^{Ah}x - x| \leq C_\alpha h^{\frac{\alpha}{2}}|x|_\alpha.$$

Then according to Lemma 4.3, we conclude that

$$\begin{aligned}
\mathbb{E}|I_1|^{2p} &\leq C_\alpha h^{\alpha p}\mathbb{E}|X_t^\varepsilon|_\alpha^{2p} \\
&\leq C_{p,T}h^{\alpha p}(1+|x|_\alpha^{2p}+|y|_\alpha^{2p}).
\end{aligned} \quad (4.14)$$

For  $I_2$ , by contractive property of the semigroup  $e^{tA}$  and Lemma 4.3, we have

$$\mathbb{E}|I_2|^{2p} \leq \mathbb{E}\left(\int_t^{t+h}|e^{(t+h-s)A}B(X_s^\varepsilon)|ds\right)^{2p}$$

$$\begin{aligned}
&\leq \mathbb{E} \left( \int_t^{t+h} |B(X_s^\varepsilon)| ds \right)^{2p} \\
&\leq C \mathbb{E} \left( \int_t^{t+h} |X_s^\varepsilon|_1^2 ds \right)^{2p} \\
&\leq C_{p,T} h^{2p} \mathbb{E} \sup_{s \leq T} |X_s^\varepsilon|_1^{4p}, \\
&\leq C_{p,T} h^{2p} (1 + |x|_\alpha^{2p} + |y|^{2p}).
\end{aligned} \tag{4.15}$$

For  $I_3$ , by Lemma 4.1, we get

$$\begin{aligned}
\mathbb{E}|I_3|^{2p} &\leq \mathbb{E} \left( \int_t^{t+h} |f(X_s^\varepsilon, Y_s^\varepsilon)| ds \right)^{2p} \\
&\leq h^{2p-1} \mathbb{E} \int_t^{t+h} |f(X_s^\varepsilon, Y_s^\varepsilon)|^{2p} ds \\
&\leq C h^{2p-1} \mathbb{E} \int_t^{t+h} (1 + |X_s^\varepsilon| + |Y_s^\varepsilon|)^{2p} ds \\
&\leq C_{p,T} h^{2p} (1 + |x|^{2p} + |y|^{2p}).
\end{aligned} \tag{4.16}$$

For  $I_4$ , notice that  $I_4 \sim N(0, S_h)$  is a Gaussian random variable, where

$$S_h x = \int_0^h e^{(h-r)A} Q_1 e^{(h-r)A^*} x dr.$$

Then for any  $p \geq 1$ , by [6, Corollary 2.17]

$$\begin{aligned}
\mathbb{E}|I_4|^{2p} &\leq C_p [\text{Tr}(S_h)]^p \\
&= C_p \left[ \sum_k \int_0^h e^{-2(h-r)\lambda_k} \alpha_k dr \right]^p \\
&\leq C_p (\sum_k \alpha_k)^p h^p.
\end{aligned} \tag{4.17}$$

Hence, (4.14) – (4.17) imply

$$\sup_{\varepsilon \in (0,1)} \mathbb{E}|X_{t+h}^\varepsilon - X_t^\varepsilon|^{2p} \leq C_{p,T} h^p (1 + |x|_\alpha^{2p} + |y|^{2p}).$$

The proof is complete.  $\square$

## 4.2 Estimates of auxiliary process $(\hat{X}_t^\varepsilon, \hat{Y}_t^\varepsilon)$

Following the idea inspired by Khasminskii in [17], we introduce an auxiliary process  $(\hat{X}_t^\varepsilon, \hat{Y}_t^\varepsilon) \in H \times H$  and divide  $[0, T]$  into intervals of size  $\delta$ , where  $\delta$  is a fixed positive number. We construct a process  $\hat{Y}_t^\varepsilon$ , with initial value  $\hat{Y}_0^\varepsilon = Y_0^\varepsilon = y$ , and for any  $t \in [k\delta, \min((k+1)\delta, T))$ ,

$$\hat{Y}_t^\varepsilon = Y_{k\delta}^\varepsilon + \frac{1}{\varepsilon} \int_{k\delta}^t A \hat{Y}_s^\varepsilon ds + \frac{1}{\varepsilon} \int_{k\delta}^t g(X_{k\delta}^\varepsilon, \hat{Y}_s^\varepsilon) ds + \frac{1}{\sqrt{\varepsilon}} \int_{k\delta}^t dW_s^{Q_2}$$

where  $(X_{k\delta}^\varepsilon, Y_{k\delta}^\varepsilon)$  are the solution of system (2.6) at time  $k\delta$ . Also, we define the process  $\hat{X}_t^\varepsilon$  by integral

$$\hat{X}_t^\varepsilon = x + \int_0^t A \hat{X}_s^\varepsilon ds + \int_0^t B(X_{s(\delta)}^\varepsilon) ds + \int_0^t f(X_{s(\delta)}^\varepsilon, \hat{Y}_s^\varepsilon) ds + W_t^{Q_1}$$

for  $t \in [0, T]$ , where  $s(\delta) = [\frac{s}{\delta}]\delta$  is the nearest breakpoint proceeding  $s$ . We remark that on each intervals the fast component  $\hat{Y}_s^\varepsilon$  does not depend on the slow component  $\hat{X}_s^\varepsilon$ , but only on the value of  $X_t^\varepsilon$  at the first point of the interval.

By the construction of  $(\hat{X}_t^\varepsilon, \hat{Y}_t^\varepsilon)$ , we can easily obtain the following estimates which will be used below. Because the proof almost follows the steps in Lemmas 4.1 and 4.3, we omit the proof here.

**Lemma 4.5** *For any  $x, y \in H$ ,  $p \geq 2$  and  $T > 0$ , there exists a constant  $C_{p,T} > 0$  such that*

$$\sup_{\varepsilon \in (0,1)} \sup_{t \in [0,T]} \mathbb{E}|\hat{Y}_t^\varepsilon|^{2p} \leq C_{p,T}(1 + |x|^{2p} + |y|^{2p}).$$

Furthermore, for any  $x \in H^\alpha, y \in H$ ,  $p \geq 2$  and  $T > 0$ , there exists a constant  $C_{p,T} > 0$  such that

$$\sup_{\varepsilon \in (0,1)} \mathbb{E}\left(\sup_{t \in [0,T]} |\hat{X}_t^\varepsilon|_\alpha^{2p}\right) \leq C_{p,T}(1 + |x|_\alpha^{2p} + |y|^{2p}).$$

Now, we will establish the the error of  $Y_t^\varepsilon - \hat{Y}_t^\varepsilon$ , and furthermore the error of  $X_t^\varepsilon - \hat{X}_t^\varepsilon$ .

**Lemma 4.6** *For any  $x \in H^\alpha, y \in H$ ,  $p \geq 2$ ,  $T > 0$  and  $\varepsilon \in (0, 1)$ , there exists a constant  $C_{p,T} > 0$  such that*

$$\sup_{0 \leq t \leq T} \mathbb{E}|Y_t^\varepsilon - \hat{Y}_t^\varepsilon|^{2p} \leq C_{p,T}(1 + |x|_\alpha^{2p} + |y|^{2p}) \frac{\delta^{p+1}}{\varepsilon}.$$

*Proof* For  $t \in [0, T]$  with  $t \in [k\delta, (k+1)\delta)$ , by Itô formula and Lemma 4.4, we have

$$\begin{aligned} \frac{d}{dt} \mathbb{E}|Y_t^\varepsilon - \hat{Y}_t^\varepsilon|^{2p} &= \frac{2p}{\varepsilon} \mathbb{E} \left[ |Y_t^\varepsilon - \hat{Y}_t^\varepsilon|^{2p-2} \langle A(Y_t^\varepsilon - \hat{Y}_t^\varepsilon), (Y_t^\varepsilon - \hat{Y}_t^\varepsilon) \rangle \right] \\ &\quad + \frac{2p}{\varepsilon} \mathbb{E} \left[ |Y_t^\varepsilon - \hat{Y}_t^\varepsilon|^{2p-2} \langle g(X_t^\varepsilon, Y_t^\varepsilon) - g(X_{k\delta}^\varepsilon, \hat{Y}_t^\varepsilon), (Y_t^\varepsilon - \hat{Y}_t^\varepsilon) \rangle \right] \\ &\leq -\frac{2p}{\varepsilon} (\lambda_1 - L_g) \mathbb{E} |Y_t^\varepsilon - \hat{Y}_t^\varepsilon|^{2p} \\ &\quad + \frac{p}{\varepsilon} (\lambda_1 - L_g) \mathbb{E} |Y_t^\varepsilon - \hat{Y}_t^\varepsilon|^{2p} + \frac{C_p}{\varepsilon} \mathbb{E} |X_t^\varepsilon - X_{k\delta}^\varepsilon|^{2p} \\ &\leq -\frac{p}{\varepsilon} (\lambda_1 - L_g) \mathbb{E} |Y_t^\varepsilon - \hat{Y}_t^\varepsilon|^{2p} + C_{p,T}(1 + |x|_\alpha^{2p} + |y|^{2p}) \frac{(t - k\delta)^p}{\varepsilon}. \end{aligned}$$

Therefore, compare theorem yields that

$$\begin{aligned} \mathbb{E}|Y_t^\varepsilon - \hat{Y}_t^\varepsilon|^{2p} &\leq \frac{C_{p,T}}{\varepsilon} (1 + |x|_\alpha^{2p} + |y|^{2p}) \int_{k\delta}^t e^{-\frac{p}{\varepsilon}(\lambda_1 - L_g)(t-s)} (s - k\delta)^p ds \\ &\leq C_{p,T}(1 + |x|_\alpha^{2p} + |y|^{2p}) \frac{\delta^{p+1}}{\varepsilon}, \end{aligned}$$

which complete the proof.  $\square$

**Lemma 4.7** *For any  $x \in H^\alpha, y \in H$ ,  $p \geq 2$ ,  $T > 0$  and  $\varepsilon \in (0, 1)$ , there exists a constant  $C_{p,T} > 0$  such that*

$$\mathbb{E} \left( \sup_{0 \leq t \leq T} |X_t^\varepsilon - \hat{X}_t^\varepsilon|^{2p} \right) \leq C_{p,T} \left( \delta^p + \frac{\delta^{p+1}}{\varepsilon} \right) (1 + |x|_\alpha^{2p} + |y|^{2p}).$$

*Proof* Recall that

$$X_t^\varepsilon = e^{tA}x + \int_0^t e^{(t-s)A}B(X_s^\varepsilon)ds + \int_0^t e^{(t-s)A}f(X_s^\varepsilon, Y_s^\varepsilon)ds + \int_0^t e^{(t-s)A}dW_s^{Q_1}$$

and

$$\hat{X}_t^\varepsilon = e^{tA}x + \int_0^t e^{(t-s)A}B(X_{s(\delta)}^\varepsilon)ds + \int_0^t e^{(t-s)A}f(X_{s(\delta)}^\varepsilon, \hat{Y}_s^\varepsilon)ds + \int_0^t e^{(t-s)A}dW_s^{Q_1}.$$

Then

$$X_t^\varepsilon - \hat{X}_t^\varepsilon = \int_0^t e^{(t-s)A}[B(X_s^\varepsilon) - B(X_{s(\delta)}^\varepsilon)]ds + \int_0^t e^{(t-s)A}[f(X_s^\varepsilon, Y_s^\varepsilon) - f(X_{s(\delta)}^\varepsilon, \hat{Y}_s^\varepsilon)]ds.$$

According to (2.5) and Lemma 2.3, we can get

$$\begin{aligned} |X_t^\varepsilon - \hat{X}_t^\varepsilon|^{2p} &\leq C_p \left\{ \int_0^t [1 + (t-s)^{-\frac{1}{2}}] |B(X_s^\varepsilon) - B(X_{s(\delta)}^\varepsilon)|_{-1} ds \right\}^{2p} \\ &\quad + C_p \left\{ \int_0^t (|X_s^\varepsilon - X_{s(\delta)}^\varepsilon| + |Y_s^\varepsilon - \hat{Y}_s^\varepsilon|) ds \right\}^{2p} \\ &\leq C_p \left\{ \int_0^t [1 + (t-s)^{-\frac{1}{2}}] |X_s^\varepsilon - X_{s(\delta)}^\varepsilon| (|X_s^\varepsilon|_1 + |X_{s(\delta)}^\varepsilon|_1) ds \right\}^{2p} \\ &\quad + C_{p,T} \int_0^t (|X_s^\varepsilon - X_{s(\delta)}^\varepsilon|^{2p} + |Y_s^\varepsilon - \hat{Y}_s^\varepsilon|^{2p}) ds \\ &\leq C_p \left\{ \int_0^t [1 + (t-s)^{-\frac{1}{2}}]^{\frac{2p}{2p-1}} ds \right\}^{2p-1} \cdot \left\{ \int_0^t |X_s^\varepsilon - X_{s(\delta)}^\varepsilon|^{4p} ds \right\}^{\frac{1}{2}} \\ &\quad \times \left( \int_0^t (|X_s^\varepsilon|_1 + |X_{s(\delta)}^\varepsilon|_1)^{4p} ds \right)^{\frac{1}{2}} \\ &\quad + C_{p,T} \int_0^t (|X_s^\varepsilon - X_{s(\delta)}^\varepsilon|^{2p} + |Y_s^\varepsilon - \hat{Y}_s^\varepsilon|^{2p}) ds \end{aligned}$$

Then, using Lemmas 4.3, 4.4 and 4.6, we obtain

$$\begin{aligned} \mathbb{E} \left( \sup_{0 \leq t \leq T} |X_t^\varepsilon - \hat{X}_t^\varepsilon|^{2p} \right) &\leq C_{p,T} \left( \int_0^T \mathbb{E} |X_s^\varepsilon - X_{s(\delta)}^\varepsilon|^{4p} ds \right)^{\frac{1}{2}} \\ &\quad \times \left( \int_0^T (\mathbb{E} |X_s^\varepsilon|_1^{4p} + \mathbb{E} |X_{s(\delta)}^\varepsilon|_1^{4p}) ds \right)^{\frac{1}{2}} \\ &\quad + C_{p,T} \int_0^T \mathbb{E} |X_s^\varepsilon - X_{s(\delta)}^\varepsilon|^{2p} + \mathbb{E} |Y_s^\varepsilon - \hat{Y}_s^\varepsilon|^{2p} ds \\ &\leq C_{p,T} (\delta^p + \frac{\delta^{p+1}}{\varepsilon}) (1 + |x|_\alpha^{2p} + |y|^{2p}). \end{aligned}$$

The proof is complete.  $\square$

### 4.3 Averaged equation

We first consider the frozen equation associate to fast motion for fixed slow component  $x \in H$ .

$$\begin{cases} \frac{\partial Y_t(\xi)}{\partial t} = AY_t(\xi) + g(x, Y_t(\xi)) + \dot{W}_t^{Q_2}, \\ Y_t(0) = Y_t(1) = 0, \quad t \in [0, \infty), \\ Y_0(\xi) = y. \end{cases} \quad (4.18)$$

Notice that  $g(x, \cdot)$  is Lipschitz continuous, it is easy to prove for any fixed slow component  $x \in H$  and any initial data  $y \in H$ , equation (4.18) has a unique mild solution  $Y_t^{x,y}$ . Let  $P_t^x$  be the transition semigroup of  $Y_t^{x,y}$ . The asymptotic behavior of  $P_t^x$  has been studied in many literatures, now we state the ergodicity for (4.18) (see [4, Theorem 3.5]).

**Theorem 4.1** *For any given value  $x, y \in H$ , there exists a unique invariant measure  $\mu^x$  for (4.18). Moreover, there exists  $C > 0$  such that for any bounded measurable function  $\varphi : H \rightarrow \mathbb{R}$ ,*

$$|P_t^x \varphi(y) - \int_H \varphi(z) \mu^x(dz)| \leq C(1 + |x| + |y|) e^{-\frac{(\lambda_1 - L_g)t}{2}} (t \wedge 1)^{-1/2} |\varphi|_\infty,$$

where  $|\varphi|_\infty = \sup_{x \in H} |\varphi(x)|$ .

Furthermore, refer to [3, Remark 3.6], we also have the following theorem:

**Theorem 4.2** *For any given value  $x, y \in H$ , there exists  $C > 0$  such that for any Lipschitz function  $\varphi : H \rightarrow \mathbb{R}$ ,*

$$|P_t^x \varphi(y) - \int_H \varphi(z) \mu^x(dz)| \leq C(1 + |x| + |y|) e^{-\frac{(\lambda_1 - L_g)t}{2}} |\varphi|_{Lip},$$

where  $|\varphi|_{Lip} = \sup_{x, y \in H} \frac{|\varphi(x) - \varphi(y)|}{|x - y|}$ .

In this section we prove that the averaging principle occurs in the sense that the slow component process  $X_t^\varepsilon$  converges strongly to the solution  $\bar{X}_t$  of the averaged equation

$$\begin{cases} d\bar{X}_t = \Delta \bar{X}_t dt + B(\bar{X}_t) dt + \bar{f}(\bar{X}_t) dt + dW_t^{Q_1}, \\ \bar{X}_0 = x. \end{cases} \quad (4.19)$$

where

$$\bar{f}(x) = \int_H f(x, y) \mu^x(dy), \quad x \in H$$

where  $\mu^x$  is the unique invariant measure for equation (4.18).

The next lemma implies the error between auxiliary process  $\hat{X}_t^\varepsilon$  and the averaging solution  $\bar{X}_t$ .

**Lemma 4.8** *For any  $x \in H^\alpha, y \in H$ ,  $p, T > 0$  and  $\varepsilon \in (0, 1)$ , then there exists positive constant  $C_{p,T}$  such that*

$$\mathbb{E} \sup_{0 \leq t \leq T} |\hat{X}_t^\varepsilon - \bar{X}_t|^{2p} \leq C_{p,T} (1 + |x|_\alpha^{2p+\frac{1}{2}} + |y|^{2p+\frac{1}{2}}) \left( \frac{1}{-\log \varepsilon} \right)^{\frac{1}{4p}}.$$

*Proof* After simple calculations, we have

$$\begin{aligned} \hat{X}_t^\varepsilon - \bar{X}_t &= \int_0^t e^{(t-s)A} [B(X_{s(\delta)}^\varepsilon) - B(\bar{X}_s)] ds + \int_0^t e^{(t-s)A} [f(X_{s(\delta)}^\varepsilon, \hat{Y}_s^\varepsilon) - \bar{f}(\bar{X}_s)] ds \\ &= \int_0^t e^{(t-s)A} [B(X_{s(\delta)}^\varepsilon) - B(\hat{X}_s^\varepsilon)] ds + \int_0^t e^{(t-s)A} [B(\hat{X}_s^\varepsilon) - B(\bar{X}_s)] ds \end{aligned}$$



$$\begin{aligned}
& + \int_0^t e^{(t-s)A} \left[ B(\hat{X}_s^\varepsilon) - B(\bar{X}_s) \right] ds + \int_0^t e^{(t-s)A} \left[ f(X_{s(\delta)}^\varepsilon, \hat{Y}_s^\varepsilon) - \bar{f}(X_s^\varepsilon) \right] ds \\
& + \int_0^t e^{(t-s)A} \left[ \bar{f}(X_s^\varepsilon) - \bar{f}(\hat{X}_s^\varepsilon) \right] ds + \int_0^t e^{(t-s)A} \left[ \bar{f}(\hat{X}_s^\varepsilon) - \bar{f}(\bar{X}_s) \right] ds \\
& := \sum_{k=1}^6 J_k(t).
\end{aligned}$$

For  $J_1(t)$ , just as the techniques in the proof of Lemma 4.7, we have

$$\begin{aligned}
\mathbb{E} \sup_{0 \leq t \leq T} |J_1(t)|^{2p} & \leq C_{p,T} \left[ \int_0^T \mathbb{E} |X_s^\varepsilon - X_{s(\delta)}^\varepsilon|^{4p} ds \right]^{\frac{1}{2}} \left[ \int_0^T (\mathbb{E} |X_s^\varepsilon|_1 + \mathbb{E} |X_{s(\delta)}^\varepsilon|_1)^{4p} ds \right]^{\frac{1}{2}} \\
& \leq C_{p,T} \delta^p (1 + |x|_\alpha^{2p} + |y|^{2p}).
\end{aligned} \tag{4.20}$$

For  $J_2(t)$ , Lemmas 4.3, 4.5 and 4.7 imply

$$\begin{aligned}
\mathbb{E} \sup_{0 \leq t \leq T} |J_2(t)|^{2p} & \leq C_{p,T} \left[ \int_0^T \mathbb{E} |X_s^\varepsilon - \hat{X}_s^\varepsilon|^{4p} ds \right]^{\frac{1}{2}} \left[ \int_0^T (\mathbb{E} |X_s^\varepsilon|_1^{4p} + \mathbb{E} |\hat{X}_s^\varepsilon|_1^{4p}) ds \right]^{\frac{1}{2}} \\
& \leq C_{p,T} (\delta^p + \frac{\delta^{p+\frac{1}{2}}}{\sqrt{\varepsilon}}) (1 + |x|_\alpha^{2p} + |y|^{2p}).
\end{aligned} \tag{4.21}$$

For  $J_3(t)$ , according to (2.5) and Lemma 2.3, we have

$$\begin{aligned}
\sup_{0 \leq t \leq T} |J_3(t)|^{2p} & \leq C_p \left\{ \sup_{0 \leq t \leq T} \int_0^t \left[ 1 + (t-s)^{-\frac{1}{2}} \right] |B(\hat{X}_s^\varepsilon) - B(\bar{X}_s)|_{-1} ds \right\}^{2p} \\
& \leq C_p \left\{ \sup_{0 \leq t \leq T} \int_0^t \left[ 1 + (t-s)^{-\frac{1}{2}} \right] |\hat{X}_s^\varepsilon - \bar{X}_s| (|\hat{X}_s^\varepsilon|_1 + |\bar{X}_s|_1) ds \right\}^{2p}.
\end{aligned} \tag{4.22}$$

In order to deal with the above estimate, we will use the skill of stopping times, i.e., for any fixed  $n \geq 1$  and  $\varepsilon > 0$ , we define the stopping time

$$\tau_n^\varepsilon = \inf \left\{ t > 0 : |\hat{X}_t^\varepsilon|_1 + |\bar{X}_t|_1 > n \right\}. \tag{4.23}$$

Then we obtain

$$\begin{aligned}
\mathbb{E} \sup_{0 \leq t \leq T \wedge \tau_n^\varepsilon} |J_3(t)|^{2p} & \leq C_p \mathbb{E} \left( \sup_{0 \leq t \leq T \wedge \tau_n^\varepsilon} \int_0^t (1 + (t-r)^{-\frac{1}{2}}) |\hat{X}_r^\varepsilon - \bar{X}_r| (|\hat{X}_r^\varepsilon|_1 + |\bar{X}_r|_1) dr \right)^{2p} \\
& \leq C_p n^{2p} \mathbb{E} \left( \sup_{0 \leq t \leq T \wedge \tau_n^\varepsilon} \int_0^t (1 + (t-r)^{-\frac{1}{2}}) |\hat{X}_r^\varepsilon - \bar{X}_r| dr \right)^{2p} \\
& \leq C_p n^{2p} \left( \sup_{0 \leq t \leq T} \int_0^t (1 + (t-r)^{-\frac{1}{2}})^{\frac{2p}{2p-1}} dr \right)^{2p-1} \mathbb{E} \int_0^{T \wedge \tau_n^\varepsilon} |\hat{X}_r^\varepsilon - \bar{X}_r|^{2p} dr \\
& \leq C_{p,T} n^{2p} \int_0^T \mathbb{E} \sup_{0 \leq r \leq s \wedge \tau_n^\varepsilon} |\hat{X}_r^\varepsilon - \bar{X}_r|^{2p} ds.
\end{aligned} \tag{4.24}$$

For  $J_5(t)$ , using the contractive property of semigroup, Lipschitz continuity of  $\bar{f}$ , and Lemma 4.7, we obtain

$$\mathbb{E} \sup_{0 \leq t \leq T} |J_5(t)|^{2p} \leq C_{p,T} \mathbb{E} \int_0^T |X_s^\varepsilon - \hat{X}_s|^{2p} ds$$

$$\leq C_{p,T}(\delta^p + \frac{\delta^{p+1}}{\varepsilon})(1 + |x|_\alpha^{2p} + |y|^{2p}). \quad (4.25)$$

For  $J_6(t)$ , similar to the method in dealing with  $J_5(t)$ , we get

$$\mathbb{E} \sup_{0 \leq t \leq T \wedge \tau_n^\varepsilon} |J_6(t)|^{2p} \leq C_{p,T} \int_0^T \mathbb{E} \sup_{0 \leq r \leq s \wedge \tau_n^\varepsilon} |\hat{X}_r^\varepsilon - \bar{X}_r|^{2p} ds. \quad (4.26)$$

Now we are going to estimate  $J_4(t)$ . For any  $t \in [0, T]$ , set  $n_t = [\frac{t}{\delta}]$ , we have  $t \in [n_t\delta, (n_t + 1)\delta \wedge T]$ . Therefore, we have representation in the form

$$J_4(t) = J_4^1(t) + J_4^2(t) + J_4^3(t),$$

where

$$J_4^1(t) = \sum_{k=0}^{n_t-1} \int_{k\delta}^{(k+1)\delta} e^{(t-s)A} \left[ f(X_{k\delta}^\varepsilon, \hat{Y}_s^\varepsilon) - \bar{f}(X_{k\delta}^\varepsilon) \right] ds,$$

$$J_4^2(t) = \sum_{k=0}^{n_t-1} \int_{k\delta}^{(k+1)\delta} e^{(t-s)A} \left[ \bar{f}(X_{k\delta}^\varepsilon) - \bar{f}(X_s^\varepsilon) \right] ds,$$

and

$$J_4^3(t) = \int_{n_t\delta}^t e^{(t-s)A} \left[ f(X_{n_t\delta}^\varepsilon, \hat{Y}_s^\varepsilon) - \bar{f}(X_s^\varepsilon) \right] ds.$$

For  $J_4^2(t)$ , we have

$$\begin{aligned} \mathbb{E} \sup_{0 \leq t \leq T} |J_4^2(t)|^{2p} &\leq C_{p,T} \int_0^T \mathbb{E} |X_{s(\delta)}^\varepsilon - X_s^\varepsilon|^{2p} ds \\ &\leq C_{p,T} \delta^p (1 + |x|_\alpha^{2p} + |y|^{2p}). \end{aligned} \quad (4.27)$$

For  $J_4^3(t)$ , by Lemmas 4.1 and 4.5 we obtain

$$\begin{aligned} \mathbb{E} \sup_{0 \leq t \leq T} |J_4^3(t)|^{2p} &\leq C_p \delta^{2p-1} \mathbb{E} \left[ \sup_{0 \leq t \leq T} \int_{n_t\delta}^t \left( 1 + |X_{n_t\delta}^\varepsilon|^{2p} + |\hat{Y}_s^\varepsilon|^{2p} + |X_s^\varepsilon|^{2p} \right) ds \right] \\ &\leq C_p \delta^{2p-1} \mathbb{E} \int_0^T \left( 1 + |X_{n_t\delta}^\varepsilon|^{2p} + |\hat{Y}_s^\varepsilon|^{2p} + |X_s^\varepsilon|^{2p} \right) ds \\ &\leq C_{p,T} \delta^{2p-1} (1 + |x|^{2p} + |y|^{2p}). \end{aligned} \quad (4.28)$$

For  $J_4^1(t)$ , by the construction of  $\hat{Y}_t^\varepsilon$  and a time shift transformation, for any fixed  $k$  and  $s \in [0, \delta]$ , we have the equalities

$$\begin{aligned} \hat{Y}_{s+k\delta}^\varepsilon &= Y_{k\delta}^\varepsilon + \frac{1}{\varepsilon} \int_{k\delta}^{k\delta+s} A \hat{Y}_r^\varepsilon dr + \frac{1}{\varepsilon} \int_{k\delta}^{k\delta+s} g(X_{k\delta}^\varepsilon, \hat{Y}_r^\varepsilon) dr + \frac{1}{\sqrt{\varepsilon}} \int_{k\delta}^{k\delta+s} dW^{Q_2}(r) \\ &= Y_{k\delta}^\varepsilon + \frac{1}{\varepsilon} \int_0^s A \hat{Y}_{r+k\delta}^\varepsilon dr + \frac{1}{\varepsilon} \int_0^s g(X_{k\delta}^\varepsilon, \hat{Y}_{r+k\delta}^\varepsilon) dr + \frac{1}{\sqrt{\varepsilon}} \int_0^s d\tilde{W}^{Q_2}(r), \end{aligned}$$

where  $\tilde{W}^{Q_2}(t) := W^{Q_2}(t + k\delta) - W^{Q_2}(k\delta)$  is the shift version of  $W^{Q_2}(t)$  and hence they have the same distribution. Let  $\bar{W}^{Q_2}(t)$  be a  $Q_2$ -Wiener process defined on the same stochastic basis and independent of  $W^{Q_1}(t)$  and  $W^{Q_2}(t)$ . We construct a process  $Y_{\frac{s}{\varepsilon}}^{X_{k\delta}^\varepsilon, Y_{k\delta}^\varepsilon}$  by means of

$$Y_{\frac{s}{\varepsilon}}^{X_{k\delta}^\varepsilon, Y_{k\delta}^\varepsilon} = Y_{k\delta}^\varepsilon + \int_0^{\frac{s}{\varepsilon}} A Y_r^{X_{k\delta}^\varepsilon, Y_{k\delta}^\varepsilon} dr + \int_0^{\frac{s}{\varepsilon}} g(X_{k\delta}^\varepsilon, Y_r^{X_{k\delta}^\varepsilon, Y_{k\delta}^\varepsilon}) dr + \int_0^{\frac{s}{\varepsilon}} d\bar{W}^{Q_2}(r)$$

$$= Y_{k\delta}^\varepsilon + \frac{1}{\varepsilon} \int_0^s AY_{\frac{r}{\varepsilon}}^{X_{k\delta}^\varepsilon, Y_{k\delta}^\varepsilon} dr + \frac{1}{\varepsilon} \int_0^s g(X_{k\delta}^\varepsilon, Y_{\frac{r}{\varepsilon}}^{X_{k\delta}^\varepsilon, Y_{k\delta}^\varepsilon}) dr + \frac{1}{\sqrt{\varepsilon}} \int_0^s d\bar{W}^{Q_2}(r),$$

where  $\bar{W}^{Q_2}(t) = \sqrt{\varepsilon} \bar{W}^{Q_2}(\frac{t}{\varepsilon})$  is the scaled version of  $\bar{W}^{Q_2}(t)$ . By the uniqueness of the solution, we have

$$(X_{k\delta}^\varepsilon, \hat{Y}_{s+k\delta}^\varepsilon) \simeq (X_{k\delta}^\varepsilon, Y_{\frac{s}{\varepsilon}}^{X_{k\delta}^\varepsilon, Y_{k\delta}^\varepsilon}),$$

where  $\simeq$  denotes a coincidence in distribution sense.

In order to estimate  $\mathbb{E} \sup_{0 \leq t \leq T} |J_4^1(t)|^{2p}$ , we firstly compute  $\mathbb{E} \sup_{0 \leq t \leq T} |J_4^1(t)|^2$ .

$$\begin{aligned} \mathbb{E} \sup_{0 \leq t \leq T} |J_4^1(t)|^2 &= \mathbb{E} \sup_{0 \leq t \leq T} \left| \sum_{k=0}^{n_t-1} e^{(t-(k+1)\delta)A} \int_{k\delta}^{(k+1)\delta} e^{((k+1)\delta-s)A} \left[ f(X_{k\delta}^\varepsilon, \hat{Y}_s^\varepsilon) - \bar{f}(X_{k\delta}^\varepsilon) \right] ds \right|^2 \\ &\leq \mathbb{E} \sup_{0 \leq t \leq T} \left\{ n_t \sum_{k=0}^{n_t-1} \left| \int_{k\delta}^{(k+1)\delta} e^{((k+1)\delta-s)A} \left[ f(X_{k\delta}^\varepsilon, \hat{Y}_s^\varepsilon) - \bar{f}(X_{k\delta}^\varepsilon) \right] ds \right|^2 \right\} \\ &\leq \left[ \frac{T}{\delta} \right] \sum_{k=0}^{\lfloor \frac{T}{\delta} \rfloor - 1} \mathbb{E} \left| \int_{k\delta}^{(k+1)\delta} e^{((k+1)\delta-s)A} \left[ f(X_{k\delta}^\varepsilon, \hat{Y}_s^\varepsilon) - \bar{f}(X_{k\delta}^\varepsilon) \right] ds \right|^2 \\ &\leq \frac{C_T}{\delta^2} \max_{0 \leq k \leq \lfloor \frac{T}{\delta} \rfloor - 1} \mathbb{E} \left| \int_{k\delta}^{(k+1)\delta} e^{((k+1)\delta-s)A} \left[ f(X_{k\delta}^\varepsilon, \hat{Y}_s^\varepsilon) - \bar{f}(X_{k\delta}^\varepsilon) \right] ds \right|^2 \\ &= C_T \frac{\varepsilon^2}{\delta^2} \max_{0 \leq k \leq \lfloor \frac{T}{\delta} \rfloor - 1} \mathbb{E} \left| \int_0^{\frac{\delta}{\varepsilon}} e^{(\delta-s\varepsilon)A} \left[ f(X_{k\delta}^\varepsilon, \hat{Y}_{s\varepsilon+k\delta}^\varepsilon) - \bar{f}(X_{k\delta}^\varepsilon) \right] ds \right|^2 \\ &= C_T \frac{\varepsilon^2}{\delta^2} \max_{0 \leq k \leq \lfloor \frac{T}{\delta} \rfloor - 1} \int_0^1 \mathbb{E} \left| \int_0^{\frac{\delta}{\varepsilon}} e^{(\delta-s\varepsilon)A} (f(X_{k\delta}^\varepsilon, \hat{Y}_{s\varepsilon+k\delta}^\varepsilon) - \bar{f}(X_{k\delta}^\varepsilon)) ds \right|^2 d\xi \\ &= C_T \frac{\varepsilon^2}{\delta^2} \max_{0 \leq k \leq \lfloor \frac{T}{\delta} \rfloor - 1} \int_0^{\frac{\delta}{\varepsilon}} \int_\tau^{\frac{\delta}{\varepsilon}} \Psi_k(s, \tau) ds d\tau, \end{aligned}$$

where

$$\begin{aligned} \Psi_k(s, \tau) &= \mathbb{E} \int_0^1 e^{(\delta-s\varepsilon)A} (f(X_{k\delta}^\varepsilon, \hat{Y}_{s\varepsilon+k\delta}^\varepsilon) - \bar{f}(X_{k\delta}^\varepsilon)) e^{(\delta-\tau\varepsilon)A} (f(X_{k\delta}^\varepsilon, \hat{Y}_{\tau\varepsilon+k\delta}^\varepsilon) - \bar{f}(X_{k\delta}^\varepsilon)) d\xi \\ &= \mathbb{E} \int_0^1 e^{(\delta-s\varepsilon)A} (f(X_{k\delta}^\varepsilon, Y_s^{X_{k\delta}^\varepsilon, Y_{k\delta}^\varepsilon}) - \bar{f}(X_{k\delta}^\varepsilon)) e^{(\delta-\tau\varepsilon)A} (f(X_{k\delta}^\varepsilon, Y_\tau^{X_{k\delta}^\varepsilon, Y_{k\delta}^\varepsilon}) - \bar{f}(X_{k\delta}^\varepsilon)) d\xi. \end{aligned}$$

Refer to [12, appendix A], there exists a constant  $C > 0$  such that

$$\begin{aligned} \Psi_k(s, \tau) &\leq C \mathbb{E} (1 + |X_{k\delta}^\varepsilon|^2 + |Y_{k\delta}^\varepsilon|^2) e^{-\frac{1}{2}(s-\tau)\eta} \\ &\leq C_T (1 + |x|^2 + |y|^2) e^{-\frac{1}{2}(s-\tau)\eta}. \end{aligned}$$

Keep in mind that we will set  $\delta = \varepsilon^{1/2}$ . Using Lemma 4.1, for any  $\varepsilon \in (0, 1)$

$$\begin{aligned} \mathbb{E} \sup_{0 \leq t \leq T} |J_4^1(t)|^2 &\leq C_T \frac{\varepsilon^2}{\delta^2} (1 + |x|^2 + |y|^2) \int_0^{\frac{\delta}{\varepsilon}} \int_\tau^{\frac{\delta}{\varepsilon}} e^{-\frac{1}{2}(s-\tau)\eta} ds d\tau \\ &= C_T \frac{\varepsilon^2}{\delta^2} (1 + |x|^2 + |y|^2) \left( \frac{2}{\eta} \cdot \frac{\delta}{\varepsilon} - \frac{4}{\eta^2} + e^{-\frac{\eta}{2} \cdot \frac{\delta}{\varepsilon}} \cdot \frac{4}{\eta^2} \right) \\ &\leq C_T \frac{\varepsilon}{\delta} (1 + |x|^2 + |y|^2). \end{aligned} \tag{4.29}$$

Besides, by Lemmas 4.3 and 4.5, we have

$$\begin{aligned}
\mathbb{E} \sup_{0 \leq t \leq T} |J_4^1(t)|^{2p} &\leq \mathbb{E} \left( \int_0^T |f(X_{[\frac{s}{\delta}] \cdot \delta}^\varepsilon, \hat{Y}_s^\varepsilon)| + \bar{f}(X_{[\frac{s}{\delta}] \cdot \delta}^\varepsilon) ds \right)^{2p} \\
&\leq C_{p,T} \left[ 1 + \sup_{s \in [0,T]} \mathbb{E}(|X_s^\varepsilon|^{2p}) + \sup_{s \in [0,T]} \mathbb{E}(|\hat{Y}_s^\varepsilon|^{2p}) \right] \\
&\leq C_{p,T} (1 + |x|^{2p} + |y|^{2p}).
\end{aligned} \tag{4.30}$$

Then, (4.29) and (4.30) imply

$$\begin{aligned}
\mathbb{E} \sup_{0 \leq t \leq T} |J_4^1(t)|^{2p} &\leq \left( \mathbb{E} \sup_{0 \leq t \leq T} |J_4^1(t)|^{2(2p-1)} \right)^{\frac{1}{2}} \left( \mathbb{E} \sup_{0 \leq t \leq T} |J_4^1(t)|^2 \right)^{\frac{1}{2}} \\
&\leq C_{p,T} (1 + |x|^{2p} + |y|^{2p}) \sqrt{\frac{\varepsilon}{\delta}}.
\end{aligned} \tag{4.31}$$

Finally, combining (4.27), (4.28) and (4.31), we get

$$\mathbb{E} \sup_{0 \leq t \leq T} |J_4(t)|^{2p} \leq C_{p,T} (1 + |x|_\alpha^{2p} + |y|^{2p}) \left( \delta^p + \delta^{2p-1} + \sqrt{\frac{\varepsilon}{\delta}} \right). \tag{4.32}$$

According to estimates (4.20), (4.21), (4.24) – (4.26), (4.32), we obtain

$$\begin{aligned}
\mathbb{E} \left( \sup_{0 \leq t \leq T \wedge \tau_n^\varepsilon} |\hat{X}_t^\varepsilon - \bar{X}_t|^{2p} \right) &\leq C_{p,T} (1 + |x|_\alpha^{2p} + |y|^{2p}) \left( \delta^p + \frac{\delta^{p+\frac{1}{2}}}{\sqrt{\varepsilon}} + \frac{\delta^{p+1}}{\varepsilon} + \delta^{2p-1} + \sqrt{\frac{\varepsilon}{\delta}} \right) \\
&\quad + C_{p,T} n^{2p} \int_0^T \mathbb{E} \sup_{0 \leq r \leq s \wedge \tau_n^\varepsilon} |\hat{X}_r^\varepsilon - \bar{X}_r|^{2p} ds.
\end{aligned}$$

Using Gronwall inequality, we get

$$\begin{aligned}
&\mathbb{E} \left( \sup_{0 \leq t \leq T \wedge \tau_n^\varepsilon} |\hat{X}_t^\varepsilon - \bar{X}_t|^{2p} \right) \\
&\leq C_{p,T} (1 + |x|_\alpha^{2p} + |y|^{2p}) \left( \delta^p + \frac{\delta^{p+\frac{1}{2}}}{\sqrt{\varepsilon}} + \frac{\delta^{p+1}}{\varepsilon} + \delta^{2p-1} + \sqrt{\frac{\varepsilon}{\delta}} \right) e^{C_{p,T} n^{2p}},
\end{aligned}$$

and this implies that

$$\begin{aligned}
&\mathbb{E} \left( \sup_{0 \leq t \leq T} |\hat{X}_t^\varepsilon - \bar{X}_t|^{2p} \cdot 1_{\{T \leq \tau_n^\varepsilon\}} \right) \\
&\leq C_{p,T} (1 + |x|_\alpha^{2p} + |y|^{2p}) \left( \delta^p + \frac{\delta^{p+\frac{1}{2}}}{\sqrt{\varepsilon}} + \frac{\delta^{p+1}}{\varepsilon} + \delta^{2p-1} + \sqrt{\frac{\varepsilon}{\delta}} \right) e^{C_{p,T} n^{2p}}.
\end{aligned}$$

Taking  $n = \sqrt[2p]{-\frac{1}{8C_{p,T}} \log \varepsilon}$ ,  $\delta = \varepsilon^{\frac{1}{2}}$ , then we get

$$\mathbb{E} \left( \sup_{0 \leq t \leq T} |\hat{X}_t^\varepsilon - \bar{X}_t|^{2p} \cdot 1_{\{T \leq \tau_n^\varepsilon\}} \right) \leq C_{p,T} \varepsilon^{\frac{1}{8}} (1 + |x|_\alpha^{2p} + |y|^{2p}). \tag{4.33}$$

On the other hand, we have

$$\mathbb{E} \left( \sup_{0 \leq t \leq T} |\hat{X}_t^\varepsilon - \bar{X}_t|^{2p} \cdot 1_{\{T > \tau_n^\varepsilon\}} \right) \leq \left( \mathbb{E} \sup_{0 \leq t \leq T} |\hat{X}_t^\varepsilon - \bar{X}_t|^{4p} \right)^{\frac{1}{2}} \cdot \left[ \mathbb{P}(T > \tau_n^\varepsilon) \right]^{\frac{1}{2}}$$

$$\begin{aligned}
&\leq C_p \left( \mathbb{E} \sup_{0 \leq t \leq T} |\hat{X}_t^\varepsilon|^{4p} + \mathbb{E} \sup_{0 \leq t \leq T} |\bar{X}_t|^{4p} \right)^{\frac{1}{2}} \\
&\quad \cdot \frac{1}{\sqrt{n}} \left( \sup_{\varepsilon \in (0,1)} \mathbb{E} \sup_{0 \leq t \leq T} |\hat{X}_t^\varepsilon|_1 + \mathbb{E} \sup_{0 \leq t \leq T} |\bar{X}_t|_1 \right)^{\frac{1}{2}} \\
&\leq \frac{C_{p,T}}{\sqrt[4p]{-\log \varepsilon}} (1 + |x|_\alpha^{2p+\frac{1}{2}} + |y|^{2p+\frac{1}{2}}), \tag{4.34}
\end{aligned}$$

where we use the fact of  $\sup_{\varepsilon \in (0,1)} \mathbb{E} \left( \sup_{0 \leq t \leq T} |\bar{X}_t^\varepsilon|_1 \right) \leq C_T(1 + |x|_\alpha)$ , which can be proved by the method similar to the proof in Lemma 4.3.

Hence, by (4.33) and (4.34), we obtain

$$\begin{aligned}
\mathbb{E} \sup_{0 \leq t \leq T} |\hat{X}_t^\varepsilon - \bar{X}_t|^{2p} &\leq C_{p,T} (1 + |x|_\alpha^{2p+\frac{1}{2}} + |y|^{2p+\frac{1}{2}}) \left\{ \varepsilon^{\frac{1}{8}} + \left( \frac{1}{-\log \varepsilon} \right)^{\frac{1}{4p}} \right\} \\
&\leq C_{p,T} (1 + |x|_\alpha^{2p+\frac{1}{2}} + |y|^{2p+\frac{1}{2}}) \left( \frac{1}{-\log \varepsilon} \right)^{\frac{1}{4p}}.
\end{aligned}$$

The proof is complete.  $\square$

## 4.4 Proof of Theorem 3.1

**Proof of Theorem 3.1:** Taking  $\delta = \varepsilon^{\frac{1}{2}}$ , then Lemma 4.7 implies

$$\begin{aligned}
\mathbb{E} \sup_{0 \leq t \leq T} |X_t^\varepsilon - \hat{X}_t^\varepsilon|^{2p} &\leq C_{p,T} (1 + |x|_\alpha^{2p} + |y|^{2p}) (\varepsilon^{\frac{p}{2}} + \varepsilon^{\frac{p}{2}-\frac{1}{2}}) \\
&\leq C_{p,T} (1 + |x|_\alpha^{2p} + |y|^{2p}) \varepsilon^{\frac{p}{2}-\frac{1}{2}}.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
\mathbb{E} \sup_{0 \leq t \leq T} |X_t^\varepsilon - \bar{X}_t|^{2p} &\leq \mathbb{E} \sup_{0 \leq t \leq T} |X_t^\varepsilon - \hat{X}_t^\varepsilon|^{2p} + \mathbb{E} \sup_{0 \leq t \leq T} |\hat{X}_t^\varepsilon - \bar{X}_t|^{2p} \\
&\leq C_{p,T} (1 + |x|_\alpha^{2p+\frac{1}{2}} + |y|^{2p+\frac{1}{2}}) \left( \frac{1}{-\log \varepsilon} \right)^{\frac{1}{4p}} \rightarrow 0 \quad (\varepsilon \rightarrow 0),
\end{aligned}$$

which completes the proof.  $\square$

## 5 Weak convergence

This section is devoted to proving Theorem 3.2 and Theorem 3.3. The idea of the proofs follows the procedure inspired by [2]. Since the proofs are tediously long and technical, a brief summary of the main ideas and steps will be provided at first. And then, every subsection will be easier to be understood. Recall that we always assume there is no noise ( $Q_1 = 0$ ) about the slow equation in system (2.6) and condition **(H4)** holds in this section.

**Step 1.** Due to the unboundedness of operator  $\Delta$ , we will use Galerkin approximation to reduce the original/infinite dimensional problem to a finite dimensional one as follows.

Let  $H_N = \text{span}\{e_k; 1 \leq k \leq N\}$  and denote  $P_N$  by the orthogonal projection of  $H$  onto  $H_N$ . Set  $f_N(x, y) = P_N(f(x, y))$ ,  $g_N(x, y) = P_N(g(x, y))$ ,  $B_N(x) = P_N(B(x))$ ,  $W_N^{Q_2}(t) =$

$P_N W^{Q_2}(t)$  for  $x, y \in H_N$ . Consider the following approximations of system (2.6) and the averaged equation (1.2):

$$\begin{cases} dX_N^\varepsilon(t) = [AX_N^\varepsilon(t) + B_N(X_N^\varepsilon(t)) + f_N(X_N^\varepsilon(t), Y_N^\varepsilon(t))]dt, \\ dY_N^\varepsilon(t) = \frac{1}{\varepsilon}[AY_N^\varepsilon(t) + g_N(X_N^\varepsilon(t), Y_N^\varepsilon(t))]dt + \frac{1}{\sqrt{\varepsilon}}dW_N^{Q_2}(t), \\ X_N^\varepsilon(0) = P_N x, Y_N^\varepsilon(0) = P_N y \end{cases} \quad (5.1)$$

and

$$\begin{cases} d\bar{X}_N(t) = [A\bar{X}_N(t) + B_N(\bar{X}_N(t)) + \bar{f}_N(\bar{X}_N(t))]dt, \\ \bar{X}_N(0) = P_N x, \end{cases} \quad (5.2)$$

where  $\bar{f}_N(x) = \int_{H_N} P_N f(x, y) \mu_N^x(dy)$ , and  $\mu_N^x(dy)$  denotes the unique invariant measure for  $dY_N(t) = [AY_N(t) + g_N(x, Y_N(t))]dt + dW_N^{Q_2}(t)$ .

For the test function  $\phi(x) \in C_b^2(H)$ , we have

$$\begin{aligned} \mathbb{E}[\phi(X^\varepsilon(t))] - \phi(\bar{X}(t)) &= \mathbb{E}[\phi(X^\varepsilon(t))] - \mathbb{E}[\phi(X_N^\varepsilon(t))] \\ &\quad + \mathbb{E}[\phi(X_N^\varepsilon(t))] - \phi(\bar{X}_N(t)) \\ &\quad + \phi(\bar{X}_N(t)) - \phi(\bar{X}(t)). \end{aligned} \quad (5.3)$$

It is not difficult to prove  $\mathbb{E}[\phi(X^\varepsilon(t))] - \mathbb{E}[\phi(X_N^\varepsilon(t))] + \phi(\bar{X}_N(t)) - \phi(\bar{X}(t)) \rightarrow 0$  as  $N \rightarrow +\infty$ . Hence, to establish Theorem 3.2 and Theorem 3.3, we only need to deal with the second term of Eq. (5.3). We will give the main idea in the next step.

**Step 2.** Inspired by [2], we will construct an asymptotic expansion of  $\mathbb{E}[\phi(X_N^\varepsilon(t))]$ . Roughly speaking, it has an expansion with respect to the small parameter  $\varepsilon$  as follows

$$\mathbb{E}[\phi(X_N^\varepsilon(t))] = \phi(\bar{X}_N(t)) + \varepsilon u_1 + v^\varepsilon.$$

In order to control the second term of Eq. (5.3), the only task left is to analyze  $u_1$  and  $v^\varepsilon$ . Almost all the work of this section is to deal with this step.

This section is organized as follows. Subsections 5.1 and 5.2 are to establish some properties of  $\bar{X}_N$  and  $(X_N^\varepsilon, Y_N^\varepsilon)$  respectively. The detailed asymptotic expansion of  $\mathbb{E}[\phi(X_N^\varepsilon(t))]$  will be given in Subsection 5.3. Based on Subsections 5.1 and 5.2, Subsection 5.4 is to study properties of  $u_1$  and  $v^\varepsilon$ . Finally, we prove Theorem 3.2 and Theorem 3.3 in Subsection 5.5.

## 5.1 Properties of $\bar{X}_N$

This subsection is to establish some properties of  $\bar{X}_N$ . **For simplicity, we will omit the index  $N$ .**

**Lemma 5.1** (1) For any  $x \in H$ ,  $t \in [0, T]$ , there exists a positive constant  $C$  such that

$$\sup_{0 \leq t \leq T} |\bar{X}_t| \leq C(1 + |x|). \quad (5.4)$$

(2) Furthermore, for any  $\gamma \in (1, \frac{3}{2})$ ,  $\theta \in (0, 1)$ ,  $x \in H^\theta$ ,  $t \in (0, T]$ , then there exists  $k \in \mathbb{N}$  such that

$$|\bar{X}_t|_\gamma \leq C(|x|_\theta + 1)t^{-\frac{\gamma-\theta}{2}}e^{C|x|^k}, \quad (5.5)$$

where  $C$  is a positive constant depending on  $\gamma, \theta, T$ .

*Proof* (1) Recall that

$$\begin{cases} \frac{d}{dt}\bar{X}_t = A\bar{X}_t + B(\bar{X}_t) + \bar{f}(\bar{X}_t), \\ \bar{X}_0 = x. \end{cases} \quad (5.6)$$

Multiply both sides of equation (5.6) by  $2\bar{X}_t$  and integrate with respect to  $\xi$ , we can easily get

$$\begin{aligned} \frac{d}{dt}|\bar{X}_t|^2 &= 2\langle A\bar{X}_t, \bar{X}_t \rangle + 2\langle \bar{f}(\bar{X}_t), \bar{X}_t \rangle \\ &\leq C(1 + |\bar{X}_t|^2), \end{aligned}$$

which implies

$$\sup_{0 \leq t \leq T} |\bar{X}_t| \leq C(1 + |x|).$$

(2) Notice that

$$\bar{X}_t = e^{tA}x + \int_0^t e^{(t-s)A}B(\bar{X}_s)ds + \int_0^t e^{(t-s)A}\bar{f}(\bar{X}_s)ds.$$

For the first term, we have

$$|e^{tA}x|_\gamma \leq Ct^{-\frac{\gamma-\theta}{2}}|x|_\theta. \quad (5.7)$$

Concerning the second term, it follows by (2.5) and Lemma 2.2 that

$$\begin{aligned} \left| \int_0^t e^{(t-s)A}B(\bar{X}_s)ds \right|_\gamma &\leq C \int_0^t \left[ 1 + (t-s)^{-\frac{1+2\gamma}{4}} \right] |B(\bar{X}_s)|_{-\frac{1}{2}} ds \\ &\leq C \int_0^t (t-s)^{-\frac{1+2\gamma}{4}} |\bar{X}_s| |\bar{X}_s|_\gamma ds \\ &\leq C \int_0^t (t-s)^{-\frac{1+2\gamma}{4}} (1 + |x|) |\bar{X}_s|_\gamma ds. \end{aligned} \quad (5.8)$$

For the last term,

$$\begin{aligned} \left| \int_0^t e^{(t-s)A}\bar{f}(\bar{X}_s)ds \right|_\gamma &\leq C \int_0^t \left[ 1 + (t-s)^{-\frac{\gamma}{2}} \right] (1 + |\bar{X}_s|) ds \\ &\leq C(1 + |x|). \end{aligned} \quad (5.9)$$

Finally, by (5.7)-(5.9) and Lemma 2.5, the conclusion follows.  $\square$

**Remark 5.1** For any  $\gamma \in (0, 1]$ ,  $\theta \in (0, 1)$ ,  $\delta \in (0, \frac{1}{2})$ ,  $t \in (0, T]$ , then by interpolation inequality, there exists  $k \in \mathbb{N}$  such that

$$|\bar{X}_t|_\gamma \leq C|\bar{X}_t|^{\frac{1+\delta-\gamma}{1+\delta}}|\bar{X}_t|^{\frac{\gamma}{1+\delta}} \leq Ct^{-\frac{1+\delta-\theta}{2}\frac{\gamma}{1+\delta}}(|x|_\theta + 1)e^{C|x|^k},$$

where  $C$  is a positive constant depending on  $\theta, \delta, T$ .

**Lemma 5.2** For any  $\theta \in (0, 1)$ ,  $\alpha \in (0, \frac{1}{2})$ ,  $x \in H^\theta$ ,  $0 < s < t \leq T$ , then there exists  $k \in \mathbb{N}$  such that

$$|\bar{X}(t, x) - \bar{X}(s, x)|_1 \leq C(t-s)^{\frac{\alpha}{2}}s^{-\frac{1+\alpha-\theta}{2}}(|x|_\theta + 1)e^{C|x|^k},$$

where  $C$  is a constant depending on  $\theta, \alpha, T$ .

*Proof* It is easy to see that

$$\bar{X}(t, x) - \bar{X}(s, x) = (e^{A(t-s)} - I) \bar{X}(s, x) + \int_s^t e^{(t-r)A} B(\bar{X}(r, x)) dr + \int_s^t e^{(t-r)A} \bar{f}(\bar{X}(r, x)) dr.$$

For the first term, by the property  $|(e^{tA} - I)x| \leq Ct^{\frac{\alpha}{2}}|x|_\alpha$  and Lemma 5.1, we have

$$\begin{aligned} |(e^{A(t-s)} - I) \bar{X}(s, x)|_1 &\leq C(t-s)^{\frac{\alpha}{2}} |\bar{X}(s, x)|_{1+\alpha} \\ &\leq C(t-s)^{\frac{\alpha}{2}} s^{-\frac{1+\alpha-\theta}{2}} (|x|_\theta + 1) e^{C|x|^k}, \end{aligned}$$

for some  $k \in \mathbb{N}$ .

For the second term, according to Lemma 5.1, we get

$$\begin{aligned} \left| \int_s^t e^{(t-r)A} B(\bar{X}(r, x)) dr \right|_1 &\leq C \int_s^t [1 + (t-r)^{-\frac{3}{4}}] |B(\bar{X}(r, x))|_{-\frac{1}{2}} dr \\ &\leq C \int_s^t [1 + (t-r)^{-\frac{3}{4}}] |\bar{X}(r, x)| |\bar{X}(r, x)|_{1+\alpha} dr \\ &\leq C \int_s^t [1 + (t-r)^{-\frac{3}{4}}] r^{-\frac{1+\alpha-\theta}{2}} (|x|_\theta + 1) e^{C|x|^k} dr \\ &\leq C(t-s)^{\frac{1}{4}} s^{-\frac{1+\alpha-\theta}{2}} (|x|_\theta + 1) e^{C|x|^k}, \end{aligned}$$

for some  $k \in \mathbb{N}$ .

For the third term, using Lemma 5.1 again, we obtain

$$\begin{aligned} \left| \int_s^t e^{(t-r)A} \bar{f}(\bar{X}(r, x)) dr \right|_1 &\leq C \int_s^t [1 + (t-r)^{-\frac{1}{2}}] |\bar{f}(\bar{X}(r, x))| dr \\ &\leq C \int_s^t [1 + (t-r)^{-\frac{1}{2}}] (1 + |\bar{X}(r, x)|) dr \\ &\leq C(t-s)^{\frac{1}{2}} (1 + |x|). \end{aligned}$$

Eventually, there exists a constant  $k \in \mathbb{N}$  such that

$$|\bar{X}(t, x) - \bar{X}(s, x)|_1 \leq C(t-s)^{\frac{\alpha}{2}} s^{-\frac{1+\alpha-\theta}{2}} (|x|_\theta + 1) e^{C|x|^k}.$$

□

**Lemma 5.3** For any  $\theta \in (0, 1)$ ,  $x \in H^\theta$ ,  $0 \leq t \leq T$ , then there exists  $k \in \mathbb{N}$  such that

$$\left| \frac{d}{dt} \bar{X}(t, x) \right| \leq Ct^{-1+\frac{\theta}{2}} (|x|_\theta^2 + 1) e^{C|x|^k},$$

where  $C$  is a constant depending on  $\theta, T$ .

*Proof* Recall that

$$\frac{d}{dt} \bar{X}(t, x) = A\bar{X}(t, x) + B(\bar{X}(t, x)) + \bar{f}(\bar{X}(t, x)).$$

Take  $\delta = \frac{1}{4}$  in Remark 5.1, then we get  $|B(\bar{X}(t, x))| \leq C|\bar{X}(t, x)|_1^2 \leq Ct^{-1+\frac{4\theta}{5}} (|x|_\theta^2 + 1) e^{C|x|^k}$  and  $|\bar{f}(\bar{X}(t, x))| \leq C(1 + |x|)$ . Hence it only remains to estimate the first term, which is estimated as follows.



Notice that

$$\begin{aligned}
\bar{X}(t, x) &= e^{tA}x + \int_0^t e^{(t-s)A} B(\bar{X}(s, x)) ds + \int_0^t e^{(t-s)A} \bar{f}(\bar{X}(s, x)) ds \\
&= e^{tA}x + \int_0^t e^{(t-s)A} B(\bar{X}(t, x)) ds + \int_0^t e^{(t-s)A} [B(\bar{X}(s, x)) - B(\bar{X}(t, x))] ds \\
&\quad + \int_0^t e^{(t-s)A} \bar{f}(\bar{X}(t, x)) ds + \int_0^t e^{(t-s)A} [\bar{f}(\bar{X}(s, x)) - \bar{f}(\bar{X}(t, x))] ds \\
&:= I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}$$

For  $I_1$ , we have

$$|AI_1| \leq Ct^{-1+\frac{\theta}{2}}|x|_\theta.$$

For  $I_2$ , we deduce from Corollary 2.1 and Remark 5.1 that

$$\begin{aligned}
|AI_2| &= |(e^{tA} - I)B(\bar{X}(t, x))| \\
&\leq 2|B(\bar{X}(t, x))| \\
&\leq 2|\bar{X}(t, x)|_1^2 \\
&\leq Ct^{-1+\frac{4\theta}{5}}(|x|_\theta^2 + 1)e^{C|x|^k}.
\end{aligned}$$

For  $I_3$ , by Lemma 5.2 and Remark 5.1, we have

$$\begin{aligned}
|AI_3| &\leq C \int_0^t \frac{1}{t-s} |B(\bar{X}(t, x)) - B(\bar{X}(s, x))| ds \\
&\leq C \int_0^t \frac{1}{t-s} |\bar{X}(t, x) - \bar{X}(s, x)|_1 (|\bar{X}(t, x)|_1 + |\bar{X}(s, x)|_1) ds \\
&\leq C \int_0^t \frac{1}{t-s} (t-s)^{\frac{\alpha}{2}} s^{-\frac{1+\alpha-\theta}{2}} (t^{-\frac{1}{2}+\frac{2\theta}{5}} + s^{-\frac{1}{2}+\frac{2\theta}{5}}) (|x|_\theta^2 + 1) e^{C|x|^k} ds \\
&\leq Ct^{-1+\frac{9\theta}{10}}(|x|_\theta^2 + 1)e^{C|x|^k}.
\end{aligned}$$

For  $I_4$ , it follows by Lemma 5.1 that

$$\begin{aligned}
|AI_4| &= |(e^{tA} - I)\bar{f}(\bar{X}(t, x))| \\
&\leq C(1 + |\bar{X}(t, x)|) \\
&\leq C(1 + |x|).
\end{aligned}$$

For  $I_5$ , by Lemma 5.2, we get

$$\begin{aligned}
|AI_5| &\leq C \int_0^t \frac{1}{t-s} |\bar{X}(t, x) - \bar{X}(s, x)| ds \\
&\leq Ct^{-\frac{1-\theta}{2}}(|x|_\theta + 1)e^{C|x|^k}.
\end{aligned}$$

The conclusion follows by the above estimates.  $\square$

The following three Lemmas deal with the derivative of  $\bar{X}(t, x)$  with respect to  $x$  in direction  $h$ . Denote  $\eta^h(t, x)$  by  $\langle D_x \bar{X}(t, x), h \rangle$ , which is the solution of the following equation

$$\begin{cases} \frac{d\eta^h(t, x)}{dt} = A\eta^h(t, x) + D\bar{f}(\bar{X}(t, x)) \cdot \eta^h(t, x) + D_\xi [\bar{X}(t, x)\eta^h(t, x)] \\ \eta^h(0, x) = h. \end{cases} \quad (5.10)$$

**Lemma 5.4** (1) For any  $t \in (0, T]$ ,  $h \in H$ , there exists a positive constant  $C$  such that

$$|\eta^h(t, x)|^2 + \int_0^t |\eta^h(s, x)|_1^2 ds \leq C e^{C|x|^5} |h|^2.$$

(2) Furthermore, for any  $\gamma \in (1, \frac{3}{2})$ ,  $\theta \in (0, 1)$ ,  $x \in H^\theta$ ,  $h \in H$ ,  $t \in (0, T]$ , there exists  $k \in \mathbb{N}$  such that

$$|\eta^h(t, x)|_\gamma \leq C t^{-\frac{\gamma}{2}} (|x|_\theta + 1) e^{C|x|^k} |h|,$$

where  $C$  is a constant depending on  $\gamma, \theta, T$ .

*Proof* Multiply both sides of the above equation (5.10) by  $\eta^h(t, x)$  and integrate with respect to  $\xi$ , then we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} |\eta^h(t, x)|^2 + |\eta^h(t, x)|_1^2 &= \int_0^1 [D\bar{f}(\bar{X}(t, x)) \eta^h(t, x)] \eta^h(t, x) d\xi \\ &\quad + \int_0^1 D_\xi [\bar{X}(t, x) \eta^h(t, x)] \eta^h(t, x) d\xi \\ &\leq C |\eta^h(t, x)|^2 - \int_0^1 \bar{X}(t, x) \eta^h(t, x) D_\xi \eta^h(t, x) d\xi \\ &= C |\eta^h(t, x)|^2 - b(\bar{X}(t, x), \eta^h(t, x), \eta^h(t, x)). \end{aligned}$$

According to Lemma 2.2 and interpolation inequality,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} |\eta^h(t, x)|^2 + |\eta^h(t, x)|_1^2 &\leq C |\eta^h(t, x)|^2 + |\bar{X}(t, x)| |\eta^h(t, x)|_1 |\eta^h(t, x)|_{\frac{3}{5}} \\ &\leq C |\eta^h(t, x)|^2 + |\bar{X}(t, x)| |\eta^h(t, x)|_1^{\frac{8}{5}} |\eta^h(t, x)|_{\frac{3}{5}}^{\frac{2}{5}} \\ &\leq C |\eta^h(t, x)|^2 + \frac{1}{2} |\eta^h(t, x)|_1^2 + C |\bar{X}(t, x)|^5 |\eta^h(t, x)|^2. \end{aligned}$$

Then, we get

$$|\eta^h(t, x)|^2 + \int_0^t |\eta^h(s, x)|_1^2 ds \leq |h|^2 + C \int_0^t (1 + |\bar{X}(s, x)|^5) |\eta^h(s, x)|^2 ds.$$

Finally, Gronwall inequality and Lemma 5.1 imply

$$|\eta^h(t, x)|^2 + \int_0^t |\eta^h(s, x)|_1^2 ds \leq C e^{C|x|^5} |h|^2.$$

(2) Notice that

$$\begin{aligned} \eta^h(t, x) &= e^{tA} h + \int_0^t e^{(t-s)A} D\bar{f}(\bar{X}(s, x)) \cdot \eta^h(s, x) ds \\ &\quad + \int_0^t e^{(t-s)A} D_\xi [\bar{X}(s, x) \eta^h(s, x)] ds. \end{aligned} \tag{5.11}$$

By (1) of Lemma 5.4, Lemma 2.2 and Lemma 5.1, it is easy to see that for  $\gamma \in (1, \frac{3}{2})$ ,

$$|\eta^h(t, x)|_\gamma \leq C t^{-\frac{\gamma}{2}} |h| + C \int_0^t (t-s)^{-\frac{\gamma}{2}} |D\bar{f}(\bar{X}(s, x)) \cdot \eta^h(s, x)| ds$$

$$\begin{aligned}
& +C \int_0^t (t-s)^{-\frac{2\gamma+1}{4}} \left| D_\xi [\bar{X}(s, x) \eta^h(s, x)] \right|_{-\frac{1}{2}} ds \\
& \leq Ct^{-\frac{\gamma}{2}} |h| + C \int_0^t (t-s)^{-\frac{\gamma}{2}} |\eta^h(s, x)| ds \\
& \quad +C \int_0^t (t-s)^{-\frac{2\gamma+1}{4}} \left| B(\bar{X}(s, x), \eta^h(s, x)) + B(\eta^h(s, x), \bar{X}(s, x)) \right|_{-\frac{1}{2}} ds \\
& \leq Ct^{-\frac{\gamma}{2}} |h| + Ce^{C|x|^5} |h| \\
& \quad +C \int_0^t (t-s)^{-\frac{2\gamma+1}{4}} \left[ |\eta^h(s, x)| |\bar{X}(s, x)|_\gamma + |\bar{X}(s, x)| |\eta^h(s, x)|_\gamma \right] ds \\
& \leq Ct^{-\frac{\gamma}{2}} |h| + Ce^{C|x|^5} |h| + Ct^{-\frac{4\gamma-3-2\theta}{4}} (|x|_\theta + 1) e^{C|x|^k} |h| \\
& \quad +C \int_0^t (t-s)^{-\frac{2\gamma+1}{4}} |\eta^h(s, x)|_\gamma (1 + |x|) ds \\
& \leq Ct^{-\frac{\gamma}{2}} (|x|_\theta + 1) e^{C|x|^k} |h| + \int_0^t (t-s)^{-\frac{2\gamma+1}{4}} |\eta^h(s, x)|_\gamma (1 + |x|) ds.
\end{aligned}$$

Consequently, by Lemma 2.5, there exists some  $k \in \mathbb{N}$  such that

$$|\eta^h(t, x)|_\gamma \leq Ct^{-\frac{\gamma}{2}} (|x|_\theta + 1) e^{C|x|^k} |h|,$$

where  $C$  is a constant depending on  $\gamma, \theta, T$ . □

**Remark 5.2** *Similar as the argument in Remark 5.1, for any  $\gamma \in (0, 1]$ ,  $t \in (0, T]$ , then there exists  $k \in \mathbb{N}$  such that*

$$|\eta^h(t, x)|_\gamma \leq Ct^{-\frac{\gamma}{2}} (|x|_\theta + 1) e^{C|x|^k} |h|,$$

where  $C$  is a constant depending on  $\theta, T$ .

**Lemma 5.5** *For any  $\alpha \in (0, \frac{1}{2})$ ,  $\theta \in (0, 1)$ ,  $x \in H^\theta$ ,  $0 < s < t \leq T$ , there exists a positive constant  $k \in \mathbb{N}$  such that*

$$|\eta^h(t, x) - \eta^h(s, x)|_1 \leq C(t-s)^{\frac{\alpha}{2}} s^{-\frac{1+\alpha}{2}} (1 + |x|_\theta) e^{C|x|^k} |h|,$$

where  $C$  is a constant depending on  $\alpha, \theta, T$ .

*Proof* It is easy to see that

$$\begin{aligned}
\eta^h(t, x) - \eta^h(s, x) &= (e^{(t-s)A} - I) \eta^h(s, x) + \int_s^t e^{(t-r)A} D\bar{f}(\bar{X}(r, x)) \cdot \eta^h(r, x) dr \\
&\quad + \int_s^t e^{(t-r)A} D_\xi [\bar{X}(r, x) \eta^h(r, x)] dr \\
&= I_1 + I_2 + I_3.
\end{aligned}$$

For  $I_1$ , we have

$$\begin{aligned}
|I_1|_1 &\leq C(t-s)^{\frac{\alpha}{2}} |\eta^h(s, x)|_{1+\alpha} \\
&\leq C(t-s)^{\frac{\alpha}{2}} s^{-\frac{1+\alpha}{2}} (1 + |x|_\theta) e^{C|x|^k} |h|.
\end{aligned}$$

Concerning  $I_2$ , it is easy to see

$$|I_2|_1 \leq \int_s^t (t-r)^{-\frac{1}{2}} |D\bar{f}(\bar{X}(r, x)) \cdot \eta^h(r, x)| dr$$

$$\leq C(t-s)^{\frac{1}{2}} e^{C|x|^5} |h|.$$

For  $I_3$ , we get

$$\begin{aligned} |I_3|_1 &\leq C \int_s^t \left[ 1 + (t-r)^{-\frac{1+\frac{1}{2}}{2}} \right] |D_\xi [\bar{X}(r, x) \eta^h(r, x)]|_{-\frac{1}{2}} dr \\ &\leq C \int_s^t \left[ 1 + (t-r)^{-\frac{3}{4}} \right] |B(\bar{X}(r, x), \eta^h(r, x)) + B(\eta^h(r, x), \bar{X}(r, x))|_{-\frac{1}{2}} dr \\ &\leq C \int_s^t \left[ 1 + (t-r)^{-\frac{3}{4}} \right] \left( |\bar{X}(r, x)| |\eta^h(r, x)|_{1+\alpha} + |\bar{X}(r, x)|_{1+\alpha} |\eta^h(r, x)| \right) dr \\ &\leq C(t-s)^{\frac{1}{4}} \left[ s^{-\frac{1+\alpha}{2}} (1 + |x|_\theta) + s^{-\frac{1+\alpha-\theta}{2}} (|x|_\theta + 1) \right] e^{C|x|^k} |h| \\ &\leq C(t-s)^{\frac{1}{4}} s^{-\frac{1+\alpha}{2}} (1 + |x|_\theta) e^{C|x|^k} |h|. \end{aligned}$$

Finally, we obtain

$$|\eta^h(t, x) - \eta^h(s, x)|_1 \leq C(t-s)^{\frac{\alpha}{2}} s^{-\frac{1+\alpha}{2}} (1 + |x|_\theta) e^{C|x|^k} |h|.$$

The proof is complete.  $\square$

**Lemma 5.6** *For any  $\theta \in (0, 1)$ ,  $x \in H^\theta$ ,  $0 < s < t \leq T$ ,  $h \in H$ , there exists  $k \in \mathbb{N}$  such that*

$$\left| \frac{d}{dt} \eta^h(t, x) \right| \leq Ct^{-1} (|x|_\theta^2 + 1) e^{C|x|^k} |h|.$$

where  $C$  is a constant depending on  $\theta, T$ .

*Proof* Recall that

$$\frac{d\eta^h(t, x)}{dt} = A\eta^h(t, x) + D\bar{f}(\bar{X}(t, x)) \cdot \eta^h(t, x) + D_\xi [\bar{X}(t, x) \eta^h(t, x)].$$

The first term is estimated as follows. Notice that

$$\begin{aligned} \eta^h(t, x) &= e^{tA} h + \int_0^t e^{(t-s)A} D\bar{f}(\bar{X}(s, x)) \cdot \eta^h(s, x) ds + \int_0^t e^{(t-s)A} D_\xi [\bar{X}(s, x) \eta^h(s, x)] ds \\ &= e^{tA} h + \int_0^t e^{(t-s)A} D\bar{f}(\bar{X}(t, x)) \cdot \eta^h(t, x) ds \\ &\quad + \int_0^t e^{(t-s)A} [D\bar{f}(\bar{X}(s, x)) \cdot \eta^h(s, x) - D\bar{f}(\bar{X}(t, x)) \cdot \eta^h(t, x)] ds \\ &\quad + \int_0^t e^{(t-s)A} D_\xi [\bar{X}(t, x) \eta^h(t, x)] ds \\ &\quad + \int_0^t e^{(t-s)A} \{ D_\xi [\bar{X}(s, x) \eta^h(s, x)] - D_\xi [\bar{X}(t, x) \eta^h(t, x)] \} ds \\ &= I_1 + I_2 + I_3 + I_4 + I_5. \end{aligned}$$

For  $I_1$ , we have

$$|Ae^{tA} h| \leq Ct^{-1} |h|. \quad (5.12)$$

For  $I_2$ , it follows by assumption **(H4)** that

$$\begin{aligned}
|AI_2| &= |(e^{tA} - I)D\bar{f}(\bar{X}(t, x)) \cdot \eta^h(t, x)| \\
&\leq 2|D\bar{f}(\bar{X}(t, x)) \cdot \eta^h(t, x)| \\
&\leq Ce^{C|x|^5}|h|.
\end{aligned} \tag{5.13}$$

For  $I_3$ , according to Lemmas 5.2 and 5.5, we obtain

$$\begin{aligned}
|AI_3| &\leq C \int_0^t \frac{1}{t-s} |D\bar{f}(\bar{X}(t, x)) \cdot \eta^h(t, x) - D\bar{f}(\bar{X}(s, x)) \cdot \eta^h(s, x)| ds \\
&\leq C \int_0^t \frac{1}{t-s} |[D\bar{f}(\bar{X}(t, x)) - D\bar{f}(\bar{X}(s, x))] \cdot \eta^h(t, x)| ds \\
&\quad + C \int_0^t \frac{1}{t-s} |D\bar{f}(\bar{X}(s, x)) \cdot (\eta^h(t, x) - \eta^h(s, x))| ds \\
&\leq C \int_0^t \frac{1}{t-s} |\bar{X}(t, x) - \bar{X}(s, x)| e^{C|x|^5} |h| ds + C \int_0^t \frac{1}{t-s} |\eta^h(t, x) - \eta^h(s, x)| ds \\
&\leq Ct^{-\frac{1}{2}}(1 + |x|_\theta) e^{C|x|^k} |h|.
\end{aligned} \tag{5.14}$$

For  $I_4$ , we deduce from Lemma 2.2, Remarks 5.1 and 5.2 that

$$\begin{aligned}
|AI_4| &= |(e^{tA} - I)D_\xi [\bar{X}(t, x)\eta^h(t, x)]| \\
&\leq 2|B(\bar{X}(t, x), \eta^h(t, x)) + B(\eta^h(t, x), \bar{X}(t, x))| \\
&\leq C|\bar{X}(t, x)|_1 |\eta^h(t, x)|_1 \\
&\leq Ct^{-1+\frac{2\theta}{5}}(|x|_\theta^2 + 1)e^{C|x|^k} |h|.
\end{aligned} \tag{5.15}$$

For  $I_5$ , by Lemmas 5.2 and 5.5, Remarks 5.1 and 5.2, we have

$$\begin{aligned}
|AI_5| &\leq C \int_0^t \frac{1}{t-s} |B(\bar{X}(s, x), \eta^h(t, x) - \eta^h(s, x)) + B(\eta^h(t, x) - \eta^h(s, x), \bar{X}(s, x)) \\
&\quad + B(\bar{X}(t, x) - \bar{X}(s, x), \eta^h(t, x)) + B(\eta^h(t, x), \bar{X}(t, x) - \bar{X}(s, x))| ds \\
&\leq C \int_0^t \frac{1}{t-s} (|\bar{X}(s, x)|_1 |\eta^h(t, x) - \eta^h(s, x)|_1 + |\eta^h(t, x)|_1 |\bar{X}(t, x) - \bar{X}(s, x)|_1) ds \\
&\leq Ct^{-1+\frac{2\theta}{5}}(|x|_\theta^2 + 1)e^{C|x|^k} |h|.
\end{aligned} \tag{5.16}$$

Then (5.12)-(5.16) imply

$$|A\eta^h(t, x)| \leq Ct^{-1}(|x|_\theta^2 + 1)e^{C|x|^k} |h|. \tag{5.17}$$

For the second term, we have

$$|D\bar{f}(\bar{X}(t, x)) \cdot \eta^h(t, x)| \leq Ce^{C|x|^5} |h|. \tag{5.18}$$

For the third term, it follows by (5.15) that

$$|D_\xi [\bar{X}(t, x)\eta^h(t, x)]| \leq Ct^{-1+\frac{2\theta}{5}}(|x|_\theta^2 + 1)e^{C|x|^k} |h|. \tag{5.19}$$

Eventually, by (5.17)-(5.19) we complete the proof.  $\square$

The following Lemma deals with the second derivative of  $\bar{X}(t, x)$  with respect to  $x$  towards directions  $h, k \in H$ . Denote  $\zeta^{h,k}(t, x)$  by  $D_{xx}^2 \bar{X}(t, x) \cdot (h, k)$ , which is the solution of the equation:

$$\begin{aligned} \frac{d\zeta^{h,k}(t, x)}{dt} &= A\zeta^{h,k}(t, x) + D^2 \bar{f}(\bar{X}(t, x)) \cdot (\eta^h(t, x), \eta^k(t, x)) + D\bar{f}(\bar{X}(t, x)) \cdot \zeta^{h,k}(t, x) \\ &\quad + D_\xi [\eta^k(t, x)\eta^h(t, x)] + D_\xi [\bar{X}(t, x)\zeta^{h,k}(t, x)]. \end{aligned} \quad (5.20)$$

**Lemma 5.7** *For any  $\theta \in (0, 1)$ ,  $x \in H^\theta$ ,  $h, k \in H$ ,  $t \leq T$ , then there exists a positive constant  $C$  such that*

$$|\zeta^{h,k}(t, x)| \leq Ce^{C|x|^5} |h||k|.$$

*Proof* Multiply both sides of the above equation (5.20) by  $\zeta^{h,k}(t, x)$ , integrate with respect to  $\xi$ , and do some elementary calculations, then we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} |\zeta^{h,k}(t, x)|^2 + |\zeta^{h,k}(t, x)|_1^2 &= \int_0^1 [D^2 \bar{f}(\bar{X}(t, x)) \cdot (\eta^h(t, x), \eta^k(t, x))] \zeta^{h,k}(t, x) d\xi \\ &\quad + \int_0^1 [D\bar{f}(\bar{X}(t, x)) \cdot \zeta^{h,k}(t, x)] \zeta^{h,k}(t, x) d\xi \\ &\quad + \int_0^1 \{D_\xi [\eta^k(t, x)\eta^h(t, x)]\} \zeta^{h,k}(t, x) d\xi \\ &\quad + \int_0^1 \{D_\xi [\bar{X}(t, x)\zeta^{h,k}(t, x)]\} \zeta^{h,k}(t, x) d\xi \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned}$$

For the first term,

$$I_1 \leq Ce^{C|x|^5} |h|^2 |k|^2 + |\zeta^{h,k}(t, x)|^2.$$

For the second term,

$$I_2 \leq C |\zeta^{h,k}(t, x)|^2.$$

For the third term, according to Lemma 2.2

$$\begin{aligned} I_3 &\leq |b(\eta^k(t, x), \eta^h(t, x), \zeta^{h,k}(t, x)) + b(\eta^h(t, x), \eta^k(t, x), \zeta^{h,k}(t, x))| \\ &\leq (|\eta^k(t, x)| |\eta^h(t, x)|_1 + |\eta^h(t, x)| |\eta^k(t, x)|_1) |\zeta^{h,k}(t, x)|_1 \\ &\leq C (|\eta^k(t, x)| |\eta^h(t, x)|_1 + |\eta^h(t, x)| |\eta^k(t, x)|_1)^2 + \frac{1}{4} |\zeta^{h,k}(t, x)|_1^2. \end{aligned}$$

For the fourth term, using integration by parts formula, Lemma 2.2 and interpolation inequality

$$\begin{aligned} I_4 &= - \int_0^1 [\bar{X}(t, x) \zeta^{h,k}(t, x)] D_\xi \zeta^{h,k}(t, x) d\xi \\ &\leq |b(\bar{X}(t, x), \zeta^{h,k}(t, x), \zeta^{h,k}(t, x))| \\ &\leq C |\bar{X}(t, x)| |\zeta^{h,k}(t, x)|_1 |\zeta^{h,k}(t, x)|_{\frac{3}{5}} \\ &\leq C |\bar{X}(t, x)| |\zeta^{h,k}(t, x)|_1^{\frac{8}{5}} |\zeta^{h,k}(t, x)|_{\frac{3}{5}}^{\frac{2}{5}} \\ &\leq C(1 + |x|^5) |\zeta^{h,k}(t, x)|^2 + \frac{1}{4} |\zeta^{h,k}(t, x)|_1^2. \end{aligned}$$

Then, by Lemma 5.4, we have

$$\begin{aligned}
|\zeta^{h,k}(t, x)|^2 &\leq C e^{C|x|^5} |h|^2 |k|^2 + C \sup_{0 \leq s \leq T} |\eta^k(s, x)|^2 \int_0^t |\eta^h(s, x)|_1^2 ds \\
&\quad + C \sup_{0 \leq s \leq T} |\eta^h(s, x)|^2 \int_0^t |\eta^k(s, x)|_1^2 ds + C \int_0^t (1 + |x|^5) |\zeta^{h,k}(s, x)|^2 ds \\
&\leq C e^{C|x|^5} |h|^2 |k|^2 + C \int_0^t (1 + |x|^5) |\zeta^{h,k}(s, x)|^2 ds.
\end{aligned}$$

Eventually, Gronwall inequality yields the desired result.  $\square$

## 5.2 Properties of $(X_N^\varepsilon, Y_N^\varepsilon)$

This subsection is to establish some properties of  $(X_N^\varepsilon, Y_N^\varepsilon)$ . **For simplicity, we will omit the index  $N$ .**

**Lemma 5.8** *The following two statements hold:*

(1) *For any  $x \in H$ ,  $t \in [0, T]$ , then there exists a positive constant  $C$  such that*

$$|X_t^\varepsilon| \leq C(1 + |x|).$$

(2) *For any  $\theta \in (0, 1)$ ,  $x \in H^\theta$ ,  $\gamma \in (1, \frac{3}{2})$ ,  $y \in H$ ,  $p \geq 1$ ,  $t \in (0, T]$ , there exists  $k \in \mathbb{N}$  such that*

$$\mathbb{E}|X_t^\varepsilon|_\gamma^p \leq C t^{-\frac{p(\gamma-\theta)}{2}} (1 + |x|_\theta^p + |y|^p) e^{C|x|^k}.$$

where  $C$  is a constant depending on  $p, \theta, \gamma, T$ .

*Proof* (1) Recall that

$$\frac{d}{dt} X_t^\varepsilon = A X_t^\varepsilon + B(X_t^\varepsilon) + f(X_t^\varepsilon, Y_t^\varepsilon).$$

Multiply both sides of the above equation by  $2X_t^\varepsilon$  and use the condition (3) in **(H4)**, then we obtain

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} |X_t^\varepsilon|^2 &= -|X_t^\varepsilon|_1^2 + \langle f(X_t^\varepsilon, Y_t^\varepsilon), X_t^\varepsilon \rangle \\
&\leq C(1 + |X_t^\varepsilon|^2).
\end{aligned}$$

Consequently, Gronwall inequality yields that  $|X_t^\varepsilon|^2 \leq C(1 + |x|^2)$ , and the result follows.

(2) Recall that

$$X_t^\varepsilon = e^{tA} x + \int_0^t e^{(t-s)A} B(X_s^\varepsilon) ds + \int_0^t e^{(t-s)A} f(X_s^\varepsilon, Y_s^\varepsilon) ds.$$

For the first term, (2.5) implies

$$|e^{tA} x|_\gamma \leq C t^{-\frac{\gamma-\theta}{2}} |x|_\theta.$$

For the second term, similar as we did in Lemma 5.1, by (1) of Lemma 5.8, we obtain

$$\left| \int_0^t e^{(t-s)A} B(X_s^\varepsilon) ds \right|_\gamma \leq C \int_0^t (t-s)^{-\frac{1+2\gamma}{4}} (1 + |x|) |X_s^\varepsilon|_\gamma ds.$$

For the last term, we get

$$\left| \int_0^t e^{(t-s)A} f(X_s^\varepsilon, Y_s^\varepsilon) ds \right|_\gamma \leq C \int_0^t (t-s)^{-\frac{\gamma}{2}} (1 + |X_s^\varepsilon| + |Y_s^\varepsilon|) ds.$$

Then by Minkowski inequality, for any  $p > 1$ , we have

$$\begin{aligned} [\mathbb{E}|X_t^\varepsilon|_\gamma^p]^{1/p} &\leq Ct^{-\frac{\gamma-\theta}{2}} |x|_\theta + C \int_0^t (t-s)^{-\frac{1+2\gamma}{4}} (1 + |x|) [\mathbb{E}|X_s^\varepsilon|_\gamma^p]^{1/p} ds \\ &\quad + C(1 + |x| + |y|). \end{aligned}$$

Hence, by Lemma 2.5, there exists  $k \in \mathbb{N}$  such that

$$[\mathbb{E}|X_t^\varepsilon|_\gamma^p]^{1/p} \leq Ct^{-\frac{\gamma-\theta}{2}} (1 + |x|_\theta + |y|) e^{C|x|^k},$$

which implies the result.  $\square$

**Remark 5.3** *Similar as the argument in Remark 5.1, for any  $\delta \in (0, \frac{1}{2})$ ,  $\theta \in (0, 1)$ ,  $x \in H^\theta$ ,  $\gamma \in (0, 1]$ ,  $y \in H$ ,  $p \geq 1$ ,  $t \in (0, T]$ , there exists  $k \in \mathbb{N}$  such that*

$$\mathbb{E}|X_t^\varepsilon|_\gamma^p \leq Ct^{-\frac{p\gamma(1+\delta-\theta)}{2(1+\delta)}} (1 + |x|_\theta^p + |y|^p) e^{C|x|^k},$$

where  $C$  is a constant depending on  $p, \delta, \theta, T$ .

**Lemma 5.9** (1) *For any  $\alpha \in (0, \frac{1}{4})$ ,  $\theta \in (0, 1)$ ,  $x \in H^\theta$ ,  $y \in H$ ,  $0 < s < t \leq T$ , there exists  $k \in \mathbb{N}$  such that*

$$[\mathbb{E}|X_t^\varepsilon - X_s^\varepsilon|_1^4]^{1/4} \leq C(t-s)^{\frac{\alpha}{2}} s^{-\frac{1+\alpha-\theta}{2}} (|x|_\theta + |y| + 1) e^{C|x|^k},$$

where  $C$  is a constant depending on  $\theta, \alpha, T$ .

(2) *For any  $\alpha \in (0, \frac{1}{4})$ ,  $x, y \in H$ ,  $0 < s < t \leq T$ , there exists a positive constant  $C$  such that*

$$\mathbb{E}|Y_t^\varepsilon - Y_s^\varepsilon|^2 \leq C(|x|^2 + |y|^2 + 1) \left[ \left( \frac{t-s}{s} \right)^{2\alpha} + \left( \frac{t-s}{\varepsilon} \right)^{2\alpha} \right].$$

*Proof* The proof of (1) is almost the same as Lemma 5.2. The proof of (2) can refer to [2, Proposition A.4], and the approach here is almost the same as the one in [2] even if we do not assume that function  $g$  is bounded. Therefore we omit the proof.  $\square$

**Lemma 5.10** *For any  $\alpha \in (0, \frac{1}{4})$ ,  $\theta \in (0, 1)$ ,  $x \in H^\theta$ ,  $0 < t \leq T$ , there exists  $k \in \mathbb{N}$  such that*

$$[\mathbb{E}|AX_t^\varepsilon|^2]^{\frac{1}{2}} \leq Ct^{-1+\frac{\theta}{2}} (|x|_\theta^2 + |y|^2 + 1) e^{C|x|^k} + C\varepsilon^{-\alpha},$$

where  $C$  is a constant depending on  $\theta, \alpha, T$ .

*Proof* For  $0 \leq s < t$ ,

$$\begin{aligned} X_t^\varepsilon &= e^{tA}x + \int_0^t e^{(t-s)A} B(X_s^\varepsilon) ds + \int_0^t e^{(t-s)A} f(X_s^\varepsilon, Y_s^\varepsilon) ds \\ &= e^{tA}x + \int_0^t e^{(t-s)A} B(X_t^\varepsilon) ds + \int_0^t e^{(t-s)A} [B(X_s^\varepsilon) - B(X_t^\varepsilon)] ds \end{aligned}$$



$$\begin{aligned}
& + \int_0^t e^{(t-s)A} f(X_s^\varepsilon, Y_s^\varepsilon) ds + \int_0^t e^{(t-s)A} [f(X_s^\varepsilon, Y_s^\varepsilon) - f(X_t^\varepsilon, Y_t^\varepsilon)] ds \\
& := I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}$$

For  $I_1$ , it is easy to see

$$|Ae^{tA}x| \leq Ct^{-1+\frac{\theta}{2}}|x|_\theta^2. \quad (5.21)$$

For  $I_2$ , taking  $p = 4$ ,  $\gamma = 1$  and  $\delta = 1/4$  in Remark 5.3, then we get

$$\begin{aligned}
[\mathbb{E}|AI_2|^2]^{1/2} &= \left[ \mathbb{E} |(e^{tA} - I)B(X_t^\varepsilon)|^2 \right]^{1/2} \\
&\leq C [\mathbb{E} |X_t^\varepsilon|_1^4]^{1/2} \\
&\leq Ct^{-1+\frac{4\theta}{5}}(1 + |x|_\theta^2 + |y|^2)e^{C|x|^k}.
\end{aligned} \quad (5.22)$$

For  $I_3$ , using Lemma 2.3, we have

$$\begin{aligned}
|AI_3| &\leq C \int_0^t \frac{1}{t-s} |B(X_s^\varepsilon) - B(X_t^\varepsilon)| ds \\
&\leq C \int_0^t \frac{1}{t-s} |X_t^\varepsilon - X_s^\varepsilon|_1 (|X_t^\varepsilon|_1 + |X_s^\varepsilon|_1) ds.
\end{aligned}$$

According to Minkowski inequality, by Lemmas 5.8 and 5.9, it follows that

$$\begin{aligned}
[\mathbb{E}|AI_3|^2]^{1/2} &\leq C \mathbb{E} \int_0^t \frac{1}{t-s} |X_t^\varepsilon - X_s^\varepsilon|_1 (|X_t^\varepsilon|_1 + |X_s^\varepsilon|_1) ds \\
&\leq C \int_0^t \frac{1}{t-s} \left\{ \mathbb{E} [|X_t^\varepsilon - X_s^\varepsilon|_1 (|X_t^\varepsilon|_1 + |X_s^\varepsilon|_1)]^2 \right\}^{1/2} ds \\
&\leq C \int_0^t \frac{1}{t-s} \left[ \mathbb{E} (|X_t^\varepsilon - X_s^\varepsilon|_1^4) \cdot (\mathbb{E} |X_t^\varepsilon|_1^4 + \mathbb{E} |X_s^\varepsilon|_1^4) \right]^{1/4} ds \\
&\leq Ct^{-1+\frac{9\theta}{10}}(|x|_\theta^2 + |y|^2 + 1)e^{C|x|^k}.
\end{aligned} \quad (5.23)$$

For  $I_4$ , we have

$$\begin{aligned}
\mathbb{E}|AI_4| &= \mathbb{E} |(e^{tA} - I)f(X_t^\varepsilon, Y_t^\varepsilon)| \\
&\leq C(1 + \mathbb{E}|X_t^\varepsilon| + \mathbb{E}|Y_t^\varepsilon|) \\
&\leq C(1 + |x| + |y|).
\end{aligned} \quad (5.24)$$

For  $I_5$ , using Minkowski inequality and Lemma 5.9, we obtain

$$\begin{aligned}
[\mathbb{E}|AI_5|^2]^{1/2} &\leq C \int_0^t \frac{1}{t-s} \left[ (\mathbb{E} |X_t^\varepsilon - X_s^\varepsilon|^2)^{\frac{1}{2}} + (\mathbb{E} |Y_t^\varepsilon - Y_s^\varepsilon|^2)^{\frac{1}{2}} \right] ds \\
&\leq Ct^{-\frac{1}{2}+\frac{\theta}{2}}(|x|_\theta + |y| + 1)e^{C|x|^k} + C\varepsilon^{-\alpha}.
\end{aligned} \quad (5.25)$$

Combining (5.21)-(5.25) yields the desired result.  $\square$

**Lemma 5.11** *For any  $\alpha \in (0, \frac{1}{4})$ ,  $\theta \in (0, 1)$ ,  $x \in H^\theta$ ,  $0 < t \leq T$ , then there exists  $k \in \mathbb{N}$  such that*

$$\mathbb{E} \left| \frac{d}{dt} X_t^\varepsilon \right| \leq Ct^{-1+\frac{\theta}{2}}(|x|_\theta^2 + |y|^2 + 1)e^{C|x|^k} + C\varepsilon^{-\alpha},$$

where  $C$  is a constant depending on  $\theta, \alpha, T$ .

*Proof* Recall that

$$\frac{d}{dt}X_t^\varepsilon = AX_t^\varepsilon + B(X_t^\varepsilon) + f(X_t^\varepsilon, Y_t^\varepsilon).$$

It is easy to see  $\mathbb{E}|B(X_t^\varepsilon)| \leq C\mathbb{E}|X_t^\varepsilon|_1^2 \leq Ct^{-1+\frac{4\theta}{5}}(|x|_\theta^2 + |y|^2 + 1)$  (taking  $p = 2, \gamma = 1$  and  $\delta = \frac{1}{4}$  in Remark 5.3) and  $\mathbb{E}|f(X_t^\varepsilon, Y_t^\varepsilon)| \leq C(1 + |x| + |y|)$ .

By Lemma 5.10, we have

$$\mathbb{E}|AX_t^\varepsilon| \leq [\mathbb{E}|AX_t^\varepsilon|^2]^{1/2} \leq Ct^{-1+\frac{\theta}{2}}(|x|_\theta^2 + |y|^2 + 1)e^{C|x|^k} + C\varepsilon^{-\alpha}.$$

The proof is complete.  $\square$

### 5.3 The asymptotic expansion of $\mathbb{E}[\phi(X_N^\varepsilon(t))]$

For any  $x, y \in H$  and  $t \geq 0$ , set

$$u_N^\varepsilon(t, x, y) = \mathbb{E}[\phi(X_N^\varepsilon(t, x, y))] \quad (5.26)$$

and

$$\bar{u}_N(t, x) = \phi(\bar{X}_N(t, x)). \quad (5.27)$$

In this subsection, we will find an expand of  $u_N^\varepsilon$  with respect to  $\varepsilon$ :

$$u_N^\varepsilon = u_0^N + \varepsilon u_1^N + v_N^\varepsilon, \quad (5.28)$$

where  $v_N^\varepsilon$  is a residual term, while  $u_0^N$  and  $u_1^N$  are constructed below, and we will claim  $u_0^N = \bar{u}_N$ .

To this end, we first introduce the following differential operators: for a  $C^2$  function  $\psi(x, y) : H \times H \rightarrow \mathbb{R}$ ,

$$L_1^N \psi(x, y) = \langle A_N x + B_N(x) + f_N(x, y), D_x \psi(x, y) \rangle,$$

$$L_2^N \psi(x, y) = \langle A_N x + g_N(x, y), D_y \psi(x, y) \rangle + \frac{1}{2} \text{Tr}(D_{yy}^2 \psi(x, y));$$

for a  $C^1$  function  $\psi : H \rightarrow \mathbb{R}$ ,

$$\bar{L}^N \psi(x) = \langle A_N x + B_N(x) + \bar{f}_N(x), D_x \psi(x) \rangle.$$

Set

$$L_N^\varepsilon = L_1^N + \frac{1}{\varepsilon} L_2^N. \quad (5.29)$$

**For simplicity, we will omit the index  $N$ .** Notice that  $\bar{u}$  does not depend on  $y$ . It is well known that  $u^\varepsilon$  and  $\bar{u}$  satisfy the following Kolmogorov equations:

$$\begin{cases} \frac{\partial u^\varepsilon(t, x, y)}{\partial t} = L^\varepsilon u^\varepsilon(t, x, y) \\ u^\varepsilon(0, x, y) = \phi(x) \end{cases} \quad (5.30)$$

and

$$\begin{cases} \frac{\partial \bar{u}(t, x)}{\partial t} = \bar{L} \bar{u}(t, x) \\ \bar{u}(0, x) = \phi(x). \end{cases}$$

By (5.29), (5.30) and (5.28), we have

$$\frac{\partial u_0}{\partial t} + \varepsilon \frac{\partial u_1}{\partial t} + \frac{\partial v^\varepsilon}{\partial t} = L_1 u_0 + \frac{1}{\varepsilon} L_2 u_0 + \varepsilon L_1 u_1 + L_2 u_1 + L_1 v^\varepsilon + \frac{1}{\varepsilon} L_2 v^\varepsilon.$$

The identification with respect to  $\varepsilon$  gives the following equations:

$$\begin{cases} L_2 u_0 = 0 \\ \frac{\partial u_0}{\partial t} = L_1 u_0 + L_2 u_1 \\ \frac{\partial v^\varepsilon}{\partial t} = L^\varepsilon v^\varepsilon + \varepsilon (L_1 u_1 - \frac{\partial u_1}{\partial t}). \end{cases} \quad (5.31)$$

To claim  $u_0$  and  $u_1$ , we need the following lemma, which is similar with [2, Lemma 4.3], but the coefficient  $f$  in our paper is Lipschitz and may be not bounded.

**Lemma 5.12** *Fix  $x \in H$ .*

(1) *If  $\Psi$  is a Lipschitz continuous function and  $\Phi$  is a function of class  $C^2$  satisfying  $L_2 \Phi = -\Psi$ , then for any  $y \in H$ , we have*

$$\Phi(y) = \int_H \Phi(z) \mu^x(dz) + \int_0^{+\infty} \mathbb{E}[\Psi(Y_s^{x,y})] ds.$$

(2) *If  $\Psi$  is a Lipschitz continuous function of class  $C^2$  such that  $\int_H \Psi(y) \mu^x(dy) = 0$ , then  $\Phi$  defined by  $\Phi(y) = \int_0^{+\infty} \mathbb{E}[\Psi(Y_s^{x,y})] ds$  is of class  $C^2$ , satisfies  $L_2 \Phi = -\Psi$ . Moreover, there exists a constant  $C$  which is independent of  $N$  such that for any  $y \in H$*

$$|\Phi(y)| \leq C(1 + |x| + |y|) |\Psi|_{Lip}. \quad (5.32)$$

*Proof* Because the proof of (1) and the first part of (2) are the same as [2, Lemma 4.3], here we only prove (5.32).

For any  $y \in H$ , Theorem 4.2 implies

$$|\mathbb{E}[\Psi(Y_s^{x,y})] - \int_H \Psi(z) \mu^x(dz)| \leq C(1 + |x| + |y|) e^{-\frac{(\lambda_1 - Lg)s}{2}} |\Psi|_{Lip}. \quad (5.33)$$

Notice that  $\int_H \Psi(y) \mu^x(dy) = 0$  and by integrating with respect to time  $s$  in (5.33), we can easily obtain (5.32), the proof is complete.  $\square$

It follows by (5.31) that function  $u_0$  is independent of  $y$ , so we can write  $u_0(t, x, y) = u_0(t, x)$ . We also choose the initial condition  $u_0(0, x) = \phi(x)$ . In view of the second equation of (5.31) and notice that  $\int_H L_2 u_1(t, x, y) \mu^x(dy) = 0$ , we have

$$\begin{aligned} \frac{\partial u_0}{\partial t}(t, x) &= \int_H \frac{\partial u_0}{\partial t}(t, x) \mu^x(dy) \\ &= \int_H L_1 u_0(t, x) \mu^x(dy) + \int_H L_2 u_1(t, x, y) \mu^x(dy) \\ &= \langle Ax + B(x) + \int_H f(x, y) \mu^x(dy), D_x u_0(t, x) \rangle \\ &= \bar{L} u_0(t, x). \end{aligned}$$

We can see that  $u_0$  and  $\bar{u}$  are solutions of the same equation, so by uniqueness of the solution we can deduce that  $u_0 = \bar{u}$ .

Then by  $\bar{L}u_0 = L_1u_0 + L_2u_1$  and definitions of  $\bar{L}$  and  $L_1$ , we have

$$\begin{aligned} L_2u_1(t, x, y) &= \langle \bar{f}(x) - f(x, y), D_xu_0(t, x) \rangle \\ &=: -\chi(t, x, y), \end{aligned}$$

where  $\chi$  is of class  $C_b^2$  with respect to  $y$ , and satisfies that for any  $t \geq 0$  and  $x \in H$ ,  $\int_H \chi(t, x, y) \mu^x(dy) = 0$ .

According to Lemma 5.12, we obtain

$$u_1(t, x, y) = \int_0^{+\infty} \mathbb{E} [\chi(t, x, Y_s^{x,y})] ds. \quad (5.34)$$

In what follows, we are going to show regularity of  $u_1$  with respect to  $t$  and  $x, y$ . In order to avoid non-integrability at the point 0, we introduce a parameter  $\rho(\varepsilon) = \varepsilon^{\frac{1}{a}}, 0 < a \leq \frac{\theta}{2}$ . By the third equation of (5.31) and Itô formula, we have

$$\begin{aligned} v^\varepsilon(t, x, y) &= \mathbb{E} [v^\varepsilon(\rho(\varepsilon), X^\varepsilon(t - \rho(\varepsilon), x, y), Y^\varepsilon(t - \rho(\varepsilon), x, y))] \\ &\quad + \varepsilon \mathbb{E} \left[ \int_{\rho(\varepsilon)}^t (L_1u_1 - \frac{\partial u_1}{\partial s})(s, X^\varepsilon(t - s, x, y), Y^\varepsilon(t - s, x, y)) ds \right]. \end{aligned}$$

By using the expansion (5.28) and  $u_0 = \bar{u}$ , then we have

$$\begin{aligned} &u^\varepsilon(t, x, y) - \bar{u}(t, x, y) \\ &= \varepsilon u_1(t, x, y) + \mathbb{E} [v^\varepsilon(\rho(\varepsilon), X^\varepsilon(t - \rho(\varepsilon), x, y), Y^\varepsilon(t - \rho(\varepsilon), x, y))] \\ &\quad + \varepsilon \mathbb{E} \left[ \int_{\rho(\varepsilon)}^t (L_1u_1 - \frac{\partial u_1}{\partial s})(s, X^\varepsilon(t - s, x, y), Y^\varepsilon(t - s, x, y)) ds \right]. \end{aligned} \quad (5.35)$$

Hence, the remaining is to give the estimates of each term in (5.35), i.e., estimate  $u_1$ ,  $v^\varepsilon$ ,  $L_1u_1$  and  $\frac{\partial u_1}{\partial t}$ , which will be proved in next subsection.

#### 5.4 Estimates of $u_1$ , $v^\varepsilon$ , $L_1u_1$ and $\frac{\partial u_1}{\partial t}$

Keep in mind that we consider Eq. (5.2) and Eq. (5.1) and the index  $N$  is omitted.

**Theorem 5.1** *There exists a positive constant  $C$  such that for any  $0 \leq t \leq T$ ,  $x, y \in H$ , we have*

$$|u_1(t, x, y)| \leq Ce^{C|x|^5} (1 + |y|).$$

*Proof* By (5.34) and Lemma 5.12, we have

$$|u_1(t, x, y)| \leq C(1 + |x| + |y|) |y \mapsto \chi(t, x, y)|_{Lip},$$

where

$$\chi(t, x, y) = \langle f(x, y) - \bar{f}(x), D_xu_0(t, x) \rangle.$$

Notice that  $f$  is Lipschitz, we can easily obtain that

$$|\chi(t, x, y_1) - \chi(t, x, y_2)| \leq C|y_1 - y_2| |D_xu_0(t, x)|.$$

Now, we are going to bound  $|D_x u_0(t, x)|$ . Recall that  $u_0 = \bar{u}$  and  $\bar{u}(t, x) = \phi(\bar{X}(t, x))$ , we have

$$D_x u_0(t, x) \cdot h = D\phi(\bar{X}(t, x)) \cdot \eta^h(t, x),$$

where

$$\eta^h(t, x) = D_x \bar{X}(t, x) \cdot h.$$

Finally the conclusion follows by Lemma 5.4.  $\square$

**Theorem 5.2** *For any  $0 \leq t \leq T$ ,  $\theta \in (0, 1]$ ,  $x \in H^\theta$ ,  $y \in H$ , then there exists  $k \in \mathbb{N}$  such that*

$$\left| \frac{\partial u_1}{\partial t}(t, x, y) \right| \leq Ct^{-1}(1 + |x|_\theta^2)e^{C|x|^k}(1 + |y|),$$

where  $C$  is a constant depending on  $\theta, T$ .

*Proof* By the definition of  $u_1$ , we have

$$\frac{\partial u_1}{\partial t}(t, x, y) = \int_0^{+\infty} \mathbb{E} \left[ \frac{\partial \chi}{\partial t}(t, x, Y_s^{x, y}) \right] ds.$$

Thanks to Lemma 5.12, we get

$$\left| \frac{\partial u_1}{\partial t}(t, x, y) \right| \leq C(1 + |x| + |y|) \left| y \mapsto \frac{\partial \chi}{\partial t}(t, x, y) \right|_{Lip},$$

where

$$\frac{\partial \chi}{\partial t}(t, x, y) = \left\langle f(x, y) - \bar{f}(x), \frac{\partial}{\partial t} D_x u_0(t, x) \right\rangle.$$

Arguing as before, we find

$$\begin{aligned} \left| \frac{\partial \chi}{\partial t}(t, x, y_1) - \frac{\partial \chi}{\partial t}(t, x, y_2) \right| &\leq |f(x, y_1) - f(x, y_2)| \left| \frac{\partial}{\partial t} D_x u_0(t, x) \right| \\ &\leq C|y_1 - y_2| \left| \frac{\partial}{\partial t} D_x u_0(t, x) \right|. \end{aligned}$$

Then for any  $h \in H$ , according to Lemmas 5.3, 5.4 and 5.6, there exists  $k \in \mathbb{N}$  such that

$$\begin{aligned} \left| \left\langle \frac{\partial}{\partial t} D_x u_0(t, x), h \right\rangle \right| &= \left| \frac{\partial}{\partial t} [D\phi(\bar{X}(t, x)) \cdot \eta^h(t, x)] \right| \\ &\leq \left| D^2\phi(\bar{X}(t, x)) \left( \eta^h(t, x), \frac{d}{dt} \bar{X}(t, x) \right) \right| + \left| D\phi(\bar{X}(t, x)) \cdot \frac{d}{dt} \eta^h(t, x) \right| \\ &\leq Ct^{-1+\frac{\theta}{2}}(1 + |x|_\theta^2)e^{C|x|^k}|h| + Ct^{-1}(1 + |x|_\theta^2)e^{C|x|^k}|h|, \end{aligned}$$

where  $C$  is a constant depending on  $\theta, T$ . Hence,

$$\left| \frac{\partial}{\partial t} D_x u_0(t, x) \right| \leq Ct^{-1}(1 + |x|_\theta^2)e^{C|x|^k}.$$

The proof is complete.  $\square$

**Theorem 5.3** *There exists a positive constant  $C$  such that for any  $0 \leq t \leq T$ ,  $x \in H^2$ ,  $y \in H$ ,*

$$|L_1 u_1(t, x, y)| \leq C e^{C|x|^5} (1 + |y|)(1 + |x|_1^2 + |y| + |Ax|).$$

*Proof* By the definition of  $L_1$ , we have

$$L_1 u_1(t, x, y) = \langle Ax + B(x) + f(x, y), D_x u_1(t, x, y) \rangle.$$

Obviously,  $|Ax + B(x) + f(x, y)| \leq C(1 + |x|_1^2 + |y| + |Ax|)$ , then we only need to estimate  $D_x u_1(t, x, y)$ . Let us recall the fact that  $u_1$  satisfies

$$L_2 u_1(t, x, y) = -\chi(t, x, y).$$

For fixed  $r > 0$ ,  $h \in H$ , define

$$\tilde{u}(t, x, y) := \frac{u_1(t, x + rh, y) - u_1(t, x, y)}{r},$$

then after doing some simple calculations, we have

$$\begin{aligned} L_2 \tilde{u}(t, x, y) &= -\frac{\chi(t, x + rh, y) - \chi(t, x, y)}{r} \\ &\quad - \left\langle \frac{g(x + rh, y) - g(x, y)}{r}, D_y u_1(t, x + rh, y) \right\rangle \\ &:= -\Gamma(t, x, y, h, r). \end{aligned}$$

According to Lemma 5.12, we obtain

$$\begin{aligned} &\frac{u_1(t, x + rh, y) - u_1(t, x, y)}{r} - \int_H \frac{u_1(t, x + rh, y) - u_1(t, x, y)}{r} \mu^x(dy) \\ &= \int_0^{+\infty} \mathbb{E}[\Gamma(t, x, Y_s^{x,y}, h, r)] ds. \end{aligned}$$

By the same argument in [2, Section 5.3], when  $r \rightarrow 0$ , we deduce that

$$\left| \lim_{r \rightarrow 0} \int_H \frac{u_1(t, x + rh, y) - u_1(t, x, y)}{r} \mu^x(dy) \right| \leq C(1 + |x|)|h|$$

and

$$\lim_{r \rightarrow 0} \Gamma(t, x, y, h, r) = \Theta(t, x, y) \cdot h := D_x \chi(t, x, y) \cdot h + \langle D_x g(x, y) \cdot h, D_y u_1(t, x, y) \rangle.$$

On one hand, it follows by the definition of  $\chi$  that

$$D_x \chi(t, x, y) \cdot h = \langle (D_x f(x, y) - D_x \bar{f}(x)) \cdot h, D_x u_0(t, x) \rangle + D_{xx}^2 u_0(t, x) \cdot (h, f(x, y)).$$

By assumption **(H4)**, Lemmas 5.4 and 5.7, we have

$$|\langle (D_x f(x, y) - D_x \bar{f}(x)) \cdot h, D_x u_0(t, x) \rangle| \leq C e^{C|x|^5} |h|$$

and

$$|D_{xx}^2 u_0(t, x) \cdot (h, k)| = |D^2 \phi(\bar{X}(t, x))(\eta^h(t, x), \eta^k(t, x)) + D\phi(\bar{X}(t, x)) \cdot \zeta^{h,k}(t, x)|$$

$$\leq Ce^{C|x|^5}|h||k|.$$

Then we obtain

$$\begin{aligned} |D_x \chi(t, x, y) \cdot h| &\leq Ce^{C|x|^5}|h| + Ce^{C|x|^5}|h| \cdot |f(x, y)| \\ &\leq Ce^{C|x|^5}(1 + |y|)|h|. \end{aligned} \quad (5.36)$$

On the other hand, by condition (3) in assumption **(H4)** and following the argument in [2, Lemma 4.3], we have

$$\begin{aligned} |\langle D_x g(x, y) \cdot h, D_y u_1(t, x, y) \rangle| &\leq C|h||D_y u_1(t, x, y)| \\ &\leq C(1 + |y|^2)|h|. \end{aligned} \quad (5.37)$$

Therefore (5.36) and (5.37) yield

$$|\Theta(t, x, y) \cdot h| \leq Ce^{C|x|^5}(1 + |y|^2)|h|. \quad (5.38)$$

Notice that for any  $t, x, r, h$ , by definition of  $\Gamma$ , we have  $\int_H \Gamma(t, x, y, h, r) \mu^x(dy) = 0$ , which implies  $\int_H \Theta(t, x, y) \cdot h \mu^x(dy) = 0$  by dominated convergence theorem. Consequently, we obtain

$$\begin{aligned} |D_x u_1(t, x, y) \cdot h| &= \lim_{r \rightarrow 0} \int_H \frac{u_1(t, x + rh, y) - u_1(t, x, y)}{r} \mu^x(dy) \\ &\quad + \int_0^\infty \mathbb{E}[\Theta(t, x, Y_s^{x, y}, h)] ds. \end{aligned}$$

According to Lemma 5.12, we don't know whether  $\Theta$  is a Lipschitz or bounded function with respect to  $y$ , and we only know it has quadratic growth by (5.38). However, the result of Lemma 5.12 can be easily extended to such function, hence we have

$$|D_x u_1(t, x, y) \cdot h| \leq Ce^{C|x|^5}(1 + |y|)|h|.$$

and therefore

$$|L_1 u_1(t, x, y)| \leq Ce^{C|x|^5}(1 + |y|)(1 + |x|_1^2 + |y| + |Ax|). \quad (5.39)$$

□

**Theorem 5.4** For any  $\alpha \in (0, \frac{1}{4})$ ,  $0 \leq t \leq T$ ,  $\theta \in (0, 1)$ ,  $x \in H^\theta$ ,  $y \in H$ , there exists  $k \in \mathbb{N}$  such that

$$|v^\varepsilon(\rho(\varepsilon), x, y)| \leq C \frac{\rho(\varepsilon)^{\frac{\theta}{2}}}{\theta} (1 + |x|_\theta^2 + |y|^2) e^{C|x|^k} + C\varepsilon e^{C|x|^5} (1 + |y|) + C\rho(\varepsilon)\varepsilon^{-\alpha},$$

where  $C$  is a constant depending on  $\alpha, \theta, T$ .

*Proof* By the asymptotic expansion  $u^\varepsilon = u_0 + \varepsilon u_1 + v^\varepsilon$ , notice that  $u_0$  is independent of  $y$ , then we can write

$$\begin{aligned} v^\varepsilon(\rho(\varepsilon), x, y) &= u^\varepsilon(\rho(\varepsilon), x, y) - u_0(\rho(\varepsilon), x) - \varepsilon u_1(\rho(\varepsilon), x, y) \\ &= [u^\varepsilon(\rho(\varepsilon), x, y) - u^\varepsilon(0, x, y)] - [u_0(\rho(\varepsilon), x) - u_0(0, x)] - \varepsilon u_1(\rho(\varepsilon), x, y) \\ &:= I_1 - I_2 - I_3. \end{aligned}$$

For  $I_1$ , by Lemma 5.11,

$$\begin{aligned}
|I_1| &= \left| \int_0^{\rho(\varepsilon)} \frac{d}{dt} u^\varepsilon(t, x, y) dt \right| \\
&= \left| \int_0^{\rho(\varepsilon)} \frac{d}{dt} \mathbb{E}[\phi(X^\varepsilon(t, x, y))] dt \right| \\
&= \left| \int_0^{\rho(\varepsilon)} \mathbb{E} \left[ D\phi(X^\varepsilon(t, x, y)) \cdot \frac{d}{dt} X^\varepsilon(t, x, y) \right] dt \right| \\
&\leq C \int_0^{\rho(\varepsilon)} \mathbb{E} \left| \frac{d}{dt} X^\varepsilon(t, x, y) \right| dt \\
&\leq C \frac{\rho(\varepsilon)^{\frac{\theta}{2}}}{\theta} (|x|_\theta^2 + |y|^2 + 1) e^{C|x|^k} + C\rho(\varepsilon)\varepsilon^{-\alpha}.
\end{aligned}$$

For  $I_2$ , recall that  $u_0 = \bar{u}$ , by Lemma 5.3,

$$\begin{aligned}
|I_2| &= \left| \int_0^{\rho(\varepsilon)} \frac{d}{dt} u_0(t, x) dt \right| \\
&= \left| \int_0^{\rho(\varepsilon)} \frac{d}{dt} \phi(\bar{X}(t, x)) dt \right| \\
&= \left| \int_0^{\rho(\varepsilon)} D\phi(\bar{X}(t, x)) \cdot \frac{d}{dt} \bar{X}(t, x) dt \right| \\
&\leq C \int_0^{\rho(\varepsilon)} \left| \frac{d}{dt} \bar{X}(t, x) \right| dt \\
&\leq \frac{C\rho(\varepsilon)^{\frac{\theta}{2}}}{\theta} (1 + |x|_\theta^2) e^{C|x|^k}.
\end{aligned}$$

For  $I_3$ , it follows by Theorem 5.1 that

$$|I_3| \leq C\varepsilon e^{C|x|^5} (1 + |y|).$$

The result follows.  $\square$

## 5.5 Proofs of Theorem 3.2 and Theorem 3.3

**Proof of Theorem 3.2:** From the expression (5.35), by Theorems 5.1-5.4, Remark 5.3, Lemma 5.10, Lemma 5.8, Hölder inequality, and notice that  $\rho(\varepsilon) = \varepsilon^{\frac{1}{a}}$ ,  $0 < a \leq \frac{\theta}{2}$ , we obtain

$$\begin{aligned}
&|\mathbb{E}[\phi(X^\varepsilon(t))] - \mathbb{E}[\phi(\bar{X}(t))]| \\
&\leq |\mathbb{E}[\phi(X^\varepsilon(t))] - \mathbb{E}[\phi(X_N^\varepsilon(t))]| + |\mathbb{E}[\phi(\bar{X}_N(t))] - \mathbb{E}[\phi(\bar{X}(t))]| + C\varepsilon e^{C|x|^5} (1 + |y|) \\
&\quad C \frac{\rho(\varepsilon)^{\frac{\theta}{2}}}{\theta} \mathbb{E} \left[ (1 + |X^\varepsilon(t - \rho(\varepsilon))|_\theta^2 + |Y^\varepsilon(t - \rho(\varepsilon))|^2) e^{C|X^\varepsilon(t - \rho(\varepsilon))|^k} \right] \\
&\quad + C\varepsilon \mathbb{E} \left[ e^{C|X^\varepsilon(t - \rho(\varepsilon))|^5} (1 + |Y^\varepsilon(t - \rho(\varepsilon))|) \right] + C\rho(\varepsilon)\varepsilon^{-\alpha} \\
&\quad + C\varepsilon \int_{\rho(\varepsilon)}^t \mathbb{E} \left[ e^{C|X^\varepsilon(t-s)|^5} (1 + |Y^\varepsilon(t-s)|) (1 + |X^\varepsilon(t-s)|_1^2 + |Y^\varepsilon(t-s)| + |AX^\varepsilon(t-s)|) \right] ds
\end{aligned}$$



$$\begin{aligned}
& +C\varepsilon \int_{\rho(\varepsilon)}^t s^{-1} \mathbb{E} \left[ (|X^\varepsilon(t-s)|_\theta^2 + 1) e^{C|X^\varepsilon(t-s)|^k} (1 + |Y^\varepsilon(t-s)|) \right] ds \\
& \leq |\mathbb{E}[\phi(X^\varepsilon(t))] - \mathbb{E}[\phi(X_N^\varepsilon(t))]| + |\mathbb{E}[\phi(\bar{X}_N(t))] - \mathbb{E}[\phi(\bar{X}(t))]| \\
& \quad + C\varepsilon + C \frac{\rho(\varepsilon)^{\frac{\theta}{2}}}{\theta} \mathbb{E}|X^\varepsilon(t - \rho(\varepsilon))|_\theta^2 + C \frac{\rho(\varepsilon)^{\frac{\theta}{2}}}{\theta} + C\varepsilon + C\rho(\varepsilon)\varepsilon^{-\alpha} \\
& \quad + C\varepsilon \int_{\rho(\varepsilon)}^t \left[ 1 + (\mathbb{E}|X^\varepsilon(t-s)|_1^4)^{\frac{1}{2}} + (\mathbb{E}|AX^\varepsilon(t-s)|^2)^{\frac{1}{2}} \right] ds \\
& \quad + C\varepsilon \int_{\rho(\varepsilon)}^t s^{-1} \left[ (\mathbb{E}|X^\varepsilon(t-s)|_\theta^4)^{1/2} + 1 \right] ds \\
& \leq |\mathbb{E}[\phi(X^\varepsilon(t))] - \mathbb{E}[\phi(X_N^\varepsilon(t))]| + |\mathbb{E}[\phi(\bar{X}_N(t))] - \mathbb{E}[\phi(\bar{X}(t))]| \\
& \quad + C\varepsilon + Ct^{-\theta+\frac{\theta^2}{1+\delta}} \rho(\varepsilon)^{\frac{\theta}{2}} - C\varepsilon \log \varepsilon \\
& \leq |\mathbb{E}[\phi(X^\varepsilon(t))] - \mathbb{E}[\phi(X_N^\varepsilon(t))]| + |\mathbb{E}[\phi(\bar{X}_N(t))] - \mathbb{E}[\phi(\bar{X}(t))]| + C\varepsilon t^{-\theta+\frac{\theta^2}{1+\delta}} - C\varepsilon \log \varepsilon,
\end{aligned}$$

where the constant  $C$  is independent of the dimension  $N$ . Then, let  $N$  go to  $+\infty$ , we get

$$|\mathbb{E}[\phi(X^\varepsilon(t))] - \mathbb{E}[\phi(\bar{X}(t))]| \leq C(1 + t^{-\theta+\frac{\theta^2}{1+\delta}})\varepsilon^{1-r}. \quad (5.40)$$

The proof is complete.  $\square$

**Proof of Theorem 3.3:** For simplicity, we only give a brief description here and state some key results benefited from a higher regularity of initial value.

Notice that if  $\theta \in (1, \frac{3}{2})$ , for any  $t \in (0, T]$ , we can easily obtain

$$|\bar{X}_t|_\theta \leq C|x|_\theta e^{C|x|}$$

and for any  $p \geq 1$ ,

$$\mathbb{E}|X_t^\varepsilon|_\theta^p \leq C(1 + |x|_\theta^p + |y|^p).$$

where the constant  $C$  is independent of  $t$ .

Finally, after doing some corresponding modifications about the results of Theorem 5.1 to Theorem 5.4, the desired result follows.  $\square$

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## References

- [1] Bogoliubov, N.N., Mitropolsky, Y.A., Asymptotic methods in the thory of Non-linear Oscillations. Gordon and Breach Science Publishers, New York (1961).
- [2] Bréhier, C.E., Strong and weak orders in averaging for SPDEs. *Stochastic Process. Appl.* 122 (2012) 2553-2593.

- [3] Cerrai, S., A Khasminskii type averaging principle for stochastic reaction-diffusion equations. *Ann. Appl. Probab.* 19 (3) (2009) 899-948.
- [4] Cerrai, S., Freidlin, M., Averaging principle for stochastic reaction-diffusion equations. *Probab. Theory Related Fields* 144 (1-2) (2009) 137-177.
- [5] Da Prato, G., Debussche, A., m-Dissipativity of Kolmogorov operators corresponding to Burgers equations with space-time white noise. *Potential Anal* 26 (2007) 31-55.
- [6] Da Prato, G., Zabczyk, J., Stochastic equations in infinite dimensions. Cambridge University Press, 1992.
- [7] Da Prato, G., Zabczyk, J., Ergodicity for Infinite Dimensional Systems, Cambridge University Press, 1996.
- [8] Da Prato, G., Debussche, A., R. Temam, Stochastic Burgers Equation. *NoDEA Nonlinear Differential Equations Appl.* 1 (1994) 389-402.
- [9] Dong, Z., Xu, T.G., One-dimensional stochastic Burgers equation driven by Lévy processes. *J. Funct. Anal.* 243 (2007) 631-678.
- [10] E, W., Khanin, K., Mazel, A., Sinai, Ya.G., Invariant measures for Burgers equation with stochastic forcing. *Ann. of Math* 151 (2000) 877-900.
- [11] Freidlin, M., Wentzell, A., Random Perturbations of Dynamical Systems, Springer, Berlin Heidelberg, 2012
- [12] Fu, H., Liu, J., Strong convergence in stochastic averaging principle for two time-scales stochastic partial differential equations. *J. Math. Anal. Appl.* 384 (1) (2011) 70-86.
- [13] Fu, H., Wan, L., Liu, J., Strong convergence in averaging principle for stochastic hyperbolic-parabolic equations with two time-scales. *Stochastic Process. Appl.* 125 (8) (2015) 3255-3279.
- [14] Fu, H., Wan, L., Wang, Y. Liu, J., Strong convergence rate in averaging principle for stochastic FitzHugh-Nagumo system with two time-scales. *J. Math. Anal. Appl.* 416 (2) (2014) 609-628.
- [15] Gonzales-Gargate, Ivan I., Ruffino, Paulo R., An averaging principle for diffusions in foliated spaces. *Ann. Probab.* 44 (1) (2016) 567-588.
- [16] Hu, B., Sun, X., Xie, Y., The Kolmogorov operator and Fokker-Planck equation associated to a stochastic Burgers equation driven by Lévy noise, *Illinois J. Math.* 58 (1) (2014) 167-205.
- [17] Khasminskii, R.Z., On an averaging principle for Itô stochastic differential equations. *Kibernetika* (4) (1968) 260-279.
- [18] Khasminskii, R.Z., Yin, G.: Limit behavior of two-time-scale diffusions revised. *J. Differential Equations* 212 (1) (2005) 85-113.
- [19] Li, X.M., An averaging principle for a completely integrable stochastic Hamiltonian system. *Nonlinearity* 21 (2008) 803-822.

- [20] Mastny, Ethan A., Haseltine, Eric L., Rawlings, James B., Two classes of quasi-steady-state model reductions for stochastic kinetics. *J.Chem.Phys* 127 (2007) 094106.
- [21] Simon, H.A., Ando, A., Aggregation of variables in dynamical systems. *Econometrica* 29 (1961) 111-138.
- [22] Truman, A., Wu, J.L., Stochastic Burgers equation with Lévy space-time white noise, in: Probabilistic Methods in Fluids, Proceedings of the Swansea 2002 Workshop, World Sci. Publishing, River Edge, NJ, (2003) 298-323.
- [23] Veretennikov, A.Y., On large deviations in averaging principle for SDEs with full dependence. *Ann.Probab.* 27 (1999) 284-296.
- [24] Volosov, V.M., Averaging in systems of ordinary differential equations. *Russ. Math. Surv.* 17 (1962) 1-126.